Lecture-10: Regenerative Processes

1 Regenerative processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be a renewal sequence, with the associated inter-renewal sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. That is, we have $S_n \triangleq \sum_{i=1}^n X_i$ for each $n \in \mathbb{N}$ and $N_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$ for each $t \in \mathbb{R}_+$.

Consider a stochastic process $Z : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ defined over the same probability space, where the nth segment of the joint process $(N, Z) : \Omega \to (\mathbb{Z}_+ \times \mathbb{R})^{\mathbb{R}_+}$ is defined as the sample path in the nth inter-renewal duration, written

$$\zeta_n \triangleq (X_n, (Z_t : t \in [S_{n-1}, S_n)), \qquad n \in \mathbb{N}.$$

Definition 1.1. The process Z is regenerative over the renewal sequence S, if its segments $(\zeta_n : n \in \mathbb{N})$ are *i.i.d.* The process Z is delayed regenerative, if S is a delayed renewal sequence and the segments of the joint process are independent with $(\zeta_n : n \geqslant 2)$ being identically distributed.

Remark 1. Let $\mathcal{F}_t \triangleq \sigma(N_u, Z_u, u \leqslant t)$ be the history until time $t \in \mathbb{R}_+$. The renewal sequence S is the **regeneration times** for the process Z, and the process Z possesses the **regenerative property** that the process $(Z_{S_{n-1}+t} : t \geqslant 0)$ is independent of history $\mathcal{F}_{S_{n-1}}$.

Remark 2. The definition says that probability law is independent of the past and shift invariant at renewal times. That is after each renewal instant, the process becomes an independent probabilistic replica of the process starting from zero.

Remark 3. If the stochastic process Z is bounded, then for any Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}[f(Z_t)|\mathcal{F}_{S_{n-1}}] = \mathbb{E}[f(Z_{t-S_{n-1}}) \mid \sigma(S_{n-1})]\mathbb{1}_{\{t \geqslant S_{n-1}\}} + f(Z_t)\mathbb{1}_{\{t < S_{n-1}\}}.$$

Example 1.2 (Age process). Let $N : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ be the renewal counting process for the renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, then the age at time t is defined as $A_t \triangleq t - S_{N_t}$. Then the age process $A : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ is regenerative. To see this, we observe that the sample path of age in nth renewal interval is given by

$$A_{S_{n-1}+t} = t, \qquad t \in [0, X_n).$$

Since the segments $(X_n, (t : t \in [0, X_n)))$ are *i.i.d.*, the result follows.

Example 1.3 (Markov chains). For a discrete time irreducible and positive recurrent homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ on finite state space $\mathcal{X} \subset \mathbb{R}$, we can inductively define the recurrent times for state $y \in \mathcal{X}$ as $\tau_y^+(0) \triangleq 0$, and

$$\tau_y^+(n) \triangleq \inf\left\{k > \tau_y^+(n-1) : X_k = y\right\}.$$

From the strong Markov property of Markov chain X, it follows that $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ is a delayed renewal sequence. For all $n \in \mathbb{N}$, we define the nth excursion time to the state y as $I_n \triangleq \left\{\tau_y^+(n-1), \dots, \tau_y^+(n) - 1\right\}$ and length of this excursion as $T_y(n) \triangleq \tau_y^+(n) - \tau_y^+(n-1)$. We show that the Markov process X is regenerative over renewal sequence τ_y^+. We can write the nth segment for the Markov chain X as $\zeta_n = (T_y(n), (X_k, k \in I_n))$. Independence of the segments follows from the strong Markov property. Further, in the segment $n \geqslant 2$ of the joint process, we can write the joint distribution for $(T_y(n), X_{\tau_y^+(n-1)+k})$ for $k < T_y(n)$ and $z \neq y$ as

$$P\left\{k < T_y(n) = m, X_{\tau_y^+(n-1)+k} = z\right\} = P_y\left\{\tau_y^+(1) > k, X_k = z\right\} P_z\left\{\tau_y^+(1) = m - k\right\}.$$

The equality follows from the strong Markov property and the homogeneity of process X.

Example 1.4 (Alternating renewal processes). A renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ where each inter-renewal duration $[S_{n-1}, S_n)$ consists of **on** time duration $[S_{n-1}, S_{n-1} + Z_n)$ followed by **off** time duration $[S_{n-1} + Z_n, S_{n-1} + Z_n + Y_n)$, is called an **alternating renewal sequence**, if $(Z, Y) : \Omega \to (\mathbb{R}_+^2)^{\mathbb{N}}$ is an *i.i.d.* random sequence. The on-time duration Z_n and off-time duration Y_n are not necessarily independent. We denote the distributions for on, off, and renewal periods by H, G, and F, respectively. Alternating renewal processes form an important class of renewal processes, and model many interesting applications.

From the definition of nth inter-renewal duration $X_n \triangleq Z_n + Y_n$, we see that $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is an *i.i.d.* sequence, and hence S is a renewal sequence. We can define an alternating stochastic process $W : \Omega \to \{0,1\}_+^{\mathbb{R}}$ that takes values 1 and 0, when the renewal process is in on and off state respectively. In particular, we can write for any time $t \in \mathbb{R}_+$

$$W_t \triangleq \mathbb{1}_{\left\{A_t < Z_{N_t+1}\right\}}.$$

For each $n \in \mathbb{N}$, we observe that $W(S_{n-1} + t) = \mathbb{1}_{t \in [0, Z_n)}$ for all $t \in [0, X_n)$. Hence, we see that the nth segment $\zeta_n = (X_n, (\mathbb{1}_{t \in [0, Z_n)} : t \in [0, X_n))$ and the segment sequence $(\zeta_n, n \in \mathbb{N})$ is *i.i.d.* , and therefore it follows that W is a regenerative process over renewal sequence S.

Example 1.5 (Age-dependent branching process). Consider a population, where each organism i lives for an *i.i.d.* random time period of $T_i : \Omega \to \mathbb{R}_+$ units with common distribution function F. Just before dying, each organism produces an *i.i.d.* random number of offsprings $N : \Omega \to \mathbb{Z}_+$, with common distribution P. Let $X(t)$ denote the number of organisms alive at time t. The stochastic process $X : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is called an **age-dependent branching process**. This is a popular model in biology for population growth of various organisms. We are interested in computing $m(t) = \mathbb{E}X(t)$ when $n = \mathbb{E}[N] = \sum_{j \in \mathbb{N}} jP_j$.

We will show that starting from an organism, the population including itself and its subsequent descendants is regenerative process. Let T_1 and N_1 denote the life time and offsprings of the first organism. If $T_1 > t$, then $X(t)$ is still equal to $X(0) = 1$. In this case, we have

$$\mathbb{E}[X(t)\mathbb{1}_{\{T_1 > t\}} \mid \sigma(T_1)] = \mathbb{E}[X(0)\mathbb{1}_{\{T_1 > t\}} \mid \sigma(T_1)] = \mathbb{1}_{\{T_1 > t\}}.$$

If $T_1 \leqslant t$, then $X(T_1) = N_1$ and each of the offsprings start the population growth, independent of the past, and stochastically identical to the population growth of the original organism starting at time T_1. Hence, we can write $X(t)\mathbb{1}_{\{T_1 \leqslant t\}} = \sum_{i=0}^{N_1} X_i(t - T_1)\mathbb{1}_{\{T_1 \leqslant t\}}$ for this case, where $(X_i(T_1 + u), u \geqslant 0)$ is a stochastic replica of $(X(u), u \geqslant 0)$, and independent for each $i \in [N_1]$. Hence for this case, we can write the following expectation conditioned on T_1

$$\mathbb{E}[X(t)\mathbb{1}_{\{T_1 \leqslant t\}} \mid \sigma(T_1)] = \mathbb{E}[\sum_{i=0}^{N_1} X_i(t - T_1)\mathbb{1}_{\{T_1 \leqslant t\}} \mid \sigma(T_1)] = nm(t - T_1)\mathbb{1}_{\{T_1 \leqslant t\}}.$$

Example 1.6 (Renewal reward process). Consider a renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with *i.i.d.* inter-renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having common distribution $F : \mathbb{R}_+ \to [0,1]$. The associated counting process is denoted by $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. We also consider an associated reward process $R : \Omega \to \mathbb{R}^{\mathbb{N}}$, such that a reward R_n is earned at the end of nth renewal interval. The reward R_n can possibly depend on inter-renewal time X_n, but is *i.i.d.* across intervals $n \in \mathbb{N}$. We can write the reward earned until time t as a random process $R : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ defined by

$$R(t) \triangleq \sum_{i=1}^{N(t)} R_i.$$

We will show that $R(t)$ and $R_{N(t)+1}$ are both regenerative processes, with regeneration intervals being the renewal intervals. For the second process, we can write the nth segment as

$$\zeta_n = (X_n, R_n).$$

It follows that the segment sequence $\zeta : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ is *i.i.d.* , and hence $R_{N(t)+1}$ is regenerative.

2 Renewal equation

Let $Z : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ be a regenerative process over renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ defined on the probability space $(\Omega, \mathcal{F}, P)$, and F be the distribution of inter-renewal times. The counting process associated with the renewal sequence S is denoted by N, and we define the history of the joint process Z, N until time t by $\mathcal{F}_t$. For any Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ and time $t \geqslant 0$, we are interested in computing time dependent marginal probability $f(t) = P\{Z_t \in A\}$. We can write the probability of the event $\{Z_t \in A\}$ by partitioning it into disjoint events as

$$P\{Z_t \in A\} = P\{Z_t \in A, S_1 > t\} + P\{Z_t \in A, S_1 \leqslant t\}.$$

We define the kernel function $K(t) = P\{S_1 > t, Z_t \in A\}$ which are typically easy to compute for any regenerative process. By the regeneration property applied at renewal instant S_1, we have

$$\mathbb{E}[\mathbb{1}_{\{Z_t \in A, S_1 \leqslant t\}} \mid \mathcal{F}_{S_1}] = \mathbb{E}[\mathbb{1}_{\{Z_{t-S_1} \in A\}} \mid \sigma(S_1)] \mathbb{1}_{\{S_1 \leqslant t\}} = f(t - S_1)\mathbb{1}_{\{S_1 \leqslant t\}}.$$

Hence, we have the following fixed point **renewal equation** for f

$$f(t) = K(t) + \int_0^t dF(s) f(t-s) = K + F * f.$$

We assume that the distribution function F and the kernel K are known, and we wish to find f, and characterize its asymptotic behavior.

Example 2.1 (Age and Excess time processes). For a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$, we can define the age process $A : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ where the age $A(t)$ at time t is the time since last renewal, i.e.

$$A(t) \triangleq t - S_{N(t)},\ t \in \mathbb{R}_+.$$

Similarly, we can define the excess time process $Y : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ where the excess time $Y(t)$ at time t is the time until next renewal, i.e.

$$Y(t) \triangleq S_{N(t)+1} - t,\ t \in \mathbb{R}_+.$$

Since the age process is regenerative for the associated renewal sequence, we can write the renewal equation for its distribution function as

$$P\{A(t) \geqslant x\} = P\{A(t) \geqslant x, S_1 > t\} + \int_0^t dF(y) P\{A(t-y) \geqslant x\}.$$

Theorem 2.2. *The renewal equation has a unique solution $f = (1+m) * K$, where $m(t) = \sum_{n \in \mathbb{N}} F_n(t)$ is the renewal function associated with the inter-renewal time distribution F.*

Proof. It follows from the renewal equation that $F * (1+m) * K = \sum_{n \in \mathbb{N}} F_n * K = m * K$. Hence, it is clear that $(1+m) * K$ is a solution to the renewal equation. For uniqueness, let f be another solution, then $h = f - K - m * K$ satisfies $h = F * h$, and hence $h = F_n * h$ for all $n \in \mathbb{N}$. From finiteness of $m(t)$, it follows that $F_n(t) \to 0$ as n grows. Hence, $\lim_{n \in \mathbb{N}} (F_n * h)(t) = 0$ for each t. □

Proposition 2.3. *Let Z be a regenerative process with state space $\mathcal{X} \subset \mathbb{R}$, over a renewal sequence S with renewal function m. For a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ and the kernel function $K(t) = P\{Z_t \in A, S_1 > t\}$, we can write for any $t \geqslant 0$*

$$P\{Z_t \in A\} = K(t) + \int_0^t dm(s) K(t-s).$$

Example 2.4 (Age and excess time processes). Since the age and excess time processes are regenerative for the associated renewal sequence, we can write the respective kernel functions K_A, K_Y

in the renewal equation for the respective distribution functions in terms of the complementary distribution function $\bar{F}$ of the inter-arrival times, as

$$K_A(t) = P\{A(t) \geqslant x, S_1 > t\} = \mathbb{1}_{\{t \geqslant x\}} \bar{F}(t), \qquad K_Y(t) = P\{Y(t) \geqslant x, S_1 > t\} = \bar{F}(t+x).$$

From the solution of renewal equation it follows that

$$P\{A(t) \geqslant x\} = \mathbb{1}_{\{t \geqslant x\}} \bar{F}(t) + \int_0^t dm(y) \mathbb{1}_{\{t-y \geqslant x\}} \bar{F}(t-y), \quad P\{Y(t) \geqslant x\} = \bar{F}(t+x) + \int_0^t dm(y) \bar{F}(t+x-y).$$

In general

Example 2.5 (Alternating renewal process). Since the alternating renewal process is regenerative for the associated renewal sequence, we can write the kernel function $K(t)$ in the renewal equation for its distribution function in terms of the complementary distribution function $\bar{F}$ of the inter-arrival times, as $K(t) = P\{W(t) = 1, S_1 > t\} = P\{H_1 > t\} = \bar{H}(t)$. From the solution of renewal equation it follows that

$$P(t) \triangleq P\{W(t) = 1\} = \bar{H}(t) + \int_0^t dm(y) \bar{H}(t-y).$$

Example 2.6 (Age-dependent branching process). Combining two cases of $\{T_1 > t\}$ and $\{T_1 \leqslant t\}$, we can write the mean function $m(t)$ as

$$m(t) = \mathbb{E}[X(t) \mathbb{1}_{\{T_1 > t\}}] + \mathbb{E}[X(t) \mathbb{1}_{\{T_1 \leqslant t\}}] = \bar{F}(t) + n \int_0^t m(t-u) dF(u). \tag{1}$$

This looks almost like a renewal function. Multiplying both sides of the above equation by $e^{-\alpha t}$, we get

$$m(t) e^{-\alpha t} = e^{-\alpha t} \bar{F}(t) + n \int_0^t e^{-\alpha(t-u)} m(t-u) e^{-\alpha u} dF(u).$$

Let $dG(t) \triangleq n e^{-\alpha t} dF(t)$, then the following choice of $\alpha > 0$ ensures that $G : \mathbb{R}_+ \to [0,1]$ is a distribution function on $\mathbb{R}_+$. In particular, let α be the unique solution to the equation

$$1 = n \int_0^\infty e^{-\alpha t} dF(t).$$

With this choice of G, the above equation (1) is a delayed renewal equation for kernel function $K(t) = e^{-\alpha t} m(t)$ with the complementary distribution of first renewal time as $\bar{H}(t) \triangleq e^{-\alpha t} \bar{F}(t)$ and the distribution of subsequent inter-renewal times being G. Let $m_G(t) \triangleq \sum_{n \in \mathbb{N}} (H * G^{(n-1)})(t)$ be the delayed renewal function associated with delayed renewal time distribution H and inter-renewal distribution G, then the solution of the equation (1) is

$$m(t) e^{-\alpha t} = e^{-\alpha t} \bar{F}(t) + \int_0^t e^{-\alpha(t-u)} \bar{F}(t-u) dm_G(u).$$