Lecture-13: Applications of Key Renewal Theorem

1 Age-dependent branching process

Consider a population, where each organism i lives for an *i.i.d.* random time period of $T_i : \Omega \to \mathbb{R}_+$ units with common distribution function F. Just before dying, each organism produces an *i.i.d.* random number of offsprings $N : \Omega \to \mathbb{Z}_+$, with common distribution P. Let X_t denote the number of organisms alive at time t. The stochastic process $X : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is called an **age-dependent branching process**. This is a popular model in biology for population growth of various organisms. We are interested in computing $m_t = \mathbb{E}X_t$ when $n = \mathbb{E}[N] = \sum_{j \in \mathbb{N}} jP_j$.

We will show that starting from an organism, the population including itself and its subsequent descendants is regenerative process. Let T_1 and N_1 denote the life time and offsprings of the first organism. If $T_1 > t$, then X_t is still equal to $X_0 = 1$. In this case, we have

$$\mathbb{E}[X_t \mathbb{1}_{\{T_1 > t\}} \mid \mathcal{F}_{T_1}] = \mathbb{E}[X_0 \mathbb{1}_{\{T_1 > t\}} \mid \mathcal{F}_{T_1}] = \mathbb{1}_{\{T_1 > t\}}.$$

If $T_1 \leqslant t$, then $X_{T_1} = N_1$ and each of the offsprings start the population growth, independent of the past, and stochastically identical to the population growth of the original organism starting at time T_1. Hence, we can write $X_t \mathbb{1}_{\{T_1 \leqslant t\}} = \sum_{i=1}^{N_1} X^i_{t-T_1} \mathbb{1}_{\{T_1 \leqslant t\}}$ for this case, where $(X^i_{T_1+u}, u \geqslant 0)$ is a stochastic replica of $(X_u, u \geqslant 0)$, and independent for each $i \in \mathbb{N}$. Hence for this case, we can write the following expectation conditioned on lifetime T_1

$$\mathbb{E}[X_t \mathbb{1}_{\{T_1 \leqslant t\}} \mid \mathcal{F}_{T_1}] = \mathbb{E}[\sum_{i=1}^{N_1} X^i_{t-T_1} \mathbb{1}_{\{T_1 \leqslant t\}} \mid \mathcal{F}_{T_1}] = n m_{t-T_1} \mathbb{1}_{\{T_1 \leqslant t\}}.$$

Combining two cases of $\{T_1 > t\}$ and $\{T_1 \leqslant t\}$, we can write the mean function m as

$$m(t) = \mathbb{E}[X_t \mathbb{1}_{\{T_1 > t\}}] + \mathbb{E}[X_t \mathbb{1}_{\{T_1 \leqslant t\}}] = \bar{F}(t) + n \int_0^t m(t-u) dF(u). \tag{1}$$

This looks almost like a renewal function.

Theorem 1.1. *The solution to* (1) *is given by*

$$m(t) = \bar{F}(t) + \int_0^t e^{\alpha u} \bar{F}(t-u) dm_G(u),$$

where $dG(t) \triangleq ne^{-\alpha t} dF(t)$, $m_g = \sum_{n \in \mathbb{N}} G^{(n)}$, *and* α *is the unique solution to*

$$1 = \int_0^\infty e^{-\alpha t} dF(t).$$

Proof. Multiplying both sides of the above equation by $e^{-\alpha t}$, we get

$$m(t) e^{-\alpha t} = e^{-\alpha t} \bar{F}(t) + n \int_0^t e^{-\alpha(t-u)} m(t-u) e^{-\alpha u} dF(u).$$

Let $dG(t) \triangleq ne^{-\alpha t} dF(t)$, then the following choice of $\alpha > 0$ ensures that $G : \mathbb{R}_+ \to [0,1]$ is a distribution function on $\mathbb{R}_+$. In particular, let α be the unique solution to the equation

$$1 = n \int_0^\infty e^{-\alpha t} dF(t).$$

With this choice of G, the above equation (1) is a delayed renewal equation for the function $f(t) = e^{-\alpha t}m(t)$ with the complementary distribution of first renewal time as $\bar{H}(t) \triangleq e^{-\alpha t}\bar{F}(t)$ and the distribution of subsequent inter-renewal times being G. That is, we have for all $t \in \mathbb{R}_+$

$$f(t) = \bar{H}(t) + \int_0^t f(t-u)dG(u).$$

That is, we have the renewal equation $f = \bar{H} + f * G$ with the delayed renewal time distribution H and inter-renewal distribution G. Let $m_G \triangleq \sum_{n\in\mathbb{N}} G^{(n)}$ be the renewal function associated with the inter-renewal time distribution G, then we observe that $f = \bar{H} + \bar{H} * m_G$ is the solution to this renewal equation (1), as we have $\bar{H} + (\bar{H} + \bar{H} * m_G) * G = \bar{H} + \bar{H} * m_G$, and therefore

$$m(t)e^{-\alpha t} = e^{-\alpha t}\bar{F}(t) + \int_0^t e^{-\alpha(t-u)}\bar{F}(t-u)dm_G(u).$$

□

Theorem 1.2. *If $X(0) = 1$, $m > 1$ and F is non lattice, then*

$$\lim_{t\to\infty} e^{-\alpha t}m(t) = \frac{n-1}{n^2\alpha\int_0^\infty xe^{-\alpha x}dF(x)},$$

where $\alpha > 0$ is the unique solution to the equation $n\int_0^\infty e^{-\alpha x}dF(x) = 1$.

Proof. Since $\bar{H}(t) = e^{-\alpha t}\bar{F}(t)$ is non-negative, monotone non-increasing, and integrable, it is directly Riemann integrable. Hence, we can apply key renewal theorem to the limiting value of solution to renewal equation to obtain

$$\lim_{t\to\infty} m(t)e^{-\alpha t} = \frac{1}{\mu_G}\int_0^\infty e^{-\alpha t}\bar{F}(t)dt = \frac{\int_0^\infty e^{-\alpha t}\bar{F}(t)dt}{n\int_0^\infty te^{-\alpha t}dF(t)}.$$

Result follows from the integration by parts,

$$\int_0^\infty e^{-\alpha t}\bar{F}(t)dt = \frac{1}{\alpha} - \frac{1}{\alpha}\int_0^\infty e^{-\alpha t}dF(t) = \frac{1}{\alpha}\left(1 - \frac{1}{n}\right).$$

□

2 Equilibrium renewal process

Recall that the limiting distribution of age for a renewal process is given by the **equilibrium distribution** $F_e : \mathbb{R}_+ \to [0,1]$ defined for an inter-renewal time distribution F as $F_e(x) = \frac{1}{\mu_F}\int_0^x \bar{F}(y)dy$ for all $x \geqslant 0$.

Lemma 2.1. *The moment generating function of $F_e(x)$ is $\tilde{F}_e(s) = \frac{1-\tilde{F}(s)}{s\mu_F}$.*

Proof. By definition, $\tilde{F}_e(s) = \mathbb{E}\left[e^{-sX}\right]$, where X is a random variable with distribution function $F_e(x)$. We use integration by parts, to write

$$\tilde{F}_e(s) = \int_0^\infty e^{-sx}dF_e(x) = \frac{1}{s\mu_F} - \frac{1}{s\mu_F}\int_0^\infty e^{-sx}dF(x) = \frac{1}{s\mu}(1 - \tilde{F}(s)).$$

□

Definition 2.2. A delayed renewal process with the initial arrival distribution $G = F_e$ is called the **equilibrium renewal process**.

Remark 1. Observe that F_e is the limiting distribution of the age and the excess time for the renewal process with common inter-renewal distribution F. Hence, if we start observing a renewal process at some arbitrarily large time t, then the observed renewal process is the equilibrium renewal process. This delayed renewal process exhibits stationary properties. That is, the limiting behaviors are exhibited for all times.

Theorem 2.3 (renewal function). *The renewal function $m^e(t)$ for the equilibrium renewal process is linear for all times. That is, $m^e(t) = \frac{t}{\mu_F}$.*

Proof. We know that the Laplace transform of renewal function $m_e(t)$ is given by

$$\tilde{m}^e(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)} = \frac{\tilde{F}_e(s)}{1-\tilde{F}(s)} = \frac{1}{s\mu_F}. \tag{2}$$

Further, we know that the Laplace transform of function t/μ is given by $\mathcal{L}_{t/\mu}(s) = \frac{1}{\mu}\int_0^\infty e^{-sx}dx = \frac{1}{s\mu}$. Since moment generating function is a one-to-one map, $m^e(t) = \frac{t}{\mu_F}$ is the unique renewal function. □

Theorem 2.4 (excess time). *The distribution of excess time $Y_e(t)$ for the equilibrium renewal process is stationary. That is,*

$$P\{Y_e(t) \leqslant x\} = F_e(x),\ t \geqslant 0. \tag{3}$$

Proof. Since the excess time $Y_e(t)$ is regenerative process and $dm^e(t) = 1/\mu_F$, we can write

$$P\{Y_e(t) > x\} = \bar{F}_e(t+x) + \frac{1}{\mu_F}\int_0^t \bar{F}(t+x-u)du = \bar{F}_e(t+x) + \frac{1}{\mu_F}\int_x^{t+x} \bar{F}(y)dy = \bar{F}_e(x).$$

□

Theorem 2.5 (Age). *The distribution of the age $A_e(t)$ for the equilibrium renewal process is stationary. That is,*

$$P\{A_e(t) \leqslant x\} = F_e(x),\ t \geqslant 0. \tag{4}$$

Proof.

$$\begin{aligned}
P\{A_e(t) \geqslant x\} &= \mathbb{1}_{\{t\geqslant x\}}\bar{F}_e(t) + \int_0^t \mathrm{d}m_e(u)\mathbb{1}_{\{t-u\geqslant x\}}\bar{F}(t-u)\\
&= \mathbb{1}_{\{t\geqslant x\}}\left(1 - \frac{1}{\mu_F}\int_0^t \bar{F}(u)\mathrm{d}u\right) + \frac{1}{\mu_F}\int_0^t \mathrm{d}u\mathbb{1}_{\{t-u\geqslant x\}}\bar{F}(t-u)\\
&= \mathbb{1}_{\{t\geqslant x\}}\left(1 - \frac{1}{\mu_F}\int_0^t \bar{F}(u)\mathrm{d}u\right) + \frac{1}{\mu_F}\int_0^{t-x} \mathrm{d}u\mathbb{1}_{\{t-u\geqslant x\}}\bar{F}(t-u)\\
&= \mathbb{1}_{\{t\geqslant x\}}\left(1 - \frac{1}{\mu_F}\int_0^t \bar{F}(u)\mathrm{d}u\right) - \frac{1}{\mu_F}\int_t^{x} \mathrm{d}u\mathbb{1}_{\{u\geqslant x\}}\bar{F}(u)\\
&= \mathbb{1}_{\{t\geqslant x\}}\left(1 - \frac{1}{\mu_F}\int_0^t \bar{F}(u)\mathrm{d}u\right) + \frac{1}{\mu_F}\int_x^{t} \mathrm{d}u\mathbb{1}_{\{t\geqslant x\}}\bar{F}(u)\\
&= \mathbb{1}_{\{t\geqslant x\}}\left(1 - \frac{1}{\mu_F}\int_0^t \bar{F}(u)\mathrm{d}u + \frac{1}{\mu_F}\int_x^{t} \mathrm{d}u\mathbb{1}_{\{t\geqslant x\}}\bar{F}(u)\right)\\
&= \mathbb{1}_{\{t\geqslant x\}}\left(1 - \frac{1}{\mu_F}\int_0^x \mathrm{d}u\mathbb{1}_{\{t\geqslant x\}}\bar{F}(u)\right)\\
&= \mathbb{1}_{\{t\geqslant x\}}(1-F_e(x))\\
&= \mathbb{1}_{\{t\geqslant x\}}\bar{F}_e(x).
\end{aligned}$$

□

Remark 2. When we start observing the counting process at time s, the observed renewal process is delayed renewal process with initial distribution Y_e at time s being identical to the distribution F_e. Hence, the number of renewals $N_e(t+s) - N_e(s)$ has the same distribution as $N_e(t)$ in duration t. That is, the distribution of counting process is shift invariant.

Theorem 2.6 (stationary increments). *The counting process $N_e : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ for the equilibrium renewal process has stationary increments.*

Proof. The Laplace transform of $N_e(t+s) - N_e(s)$ is identical to Laplace transform of $N_e(t)$. Result holds from the uniqueness of inverse of Laplace transforms. □

Example 2.7 (Poisson process). Consider the case, when inter-renewal time distribution F for a delay renewal process is exponential with rate λ. Here, one would expect the equilibrium distribution $F_e = F$, since Poisson process has stationary and independent increments. We observe that

$$F_e(x) = \frac{1}{\mu}\int_0^x \bar{F}(y)dy = \lambda \int_0^x \mathrm{e}^{-\lambda y} dy = 1 - e^{-\lambda x} = F(x).$$

We see that F_e is also distributed exponentially with rate λ. Indeed, this is a Poisson process with rate λ.