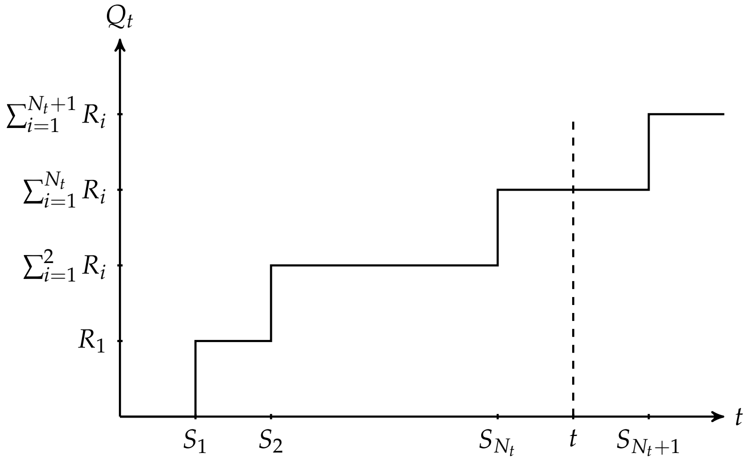
Lecture-14: Renewal Reward Processes

1 Renewal reward process

Definition 1.1. Consider a counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, where the *i.i.d.* inter-renewal time sequence is denoted by $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having common distribution F. At the end of each renewal interval $n \in \mathbb{N}$, a random reward $R_n : \Omega \to \mathbb{R}$ is earned at time S_n, where the reward R_n is possibly dependent on the duration X_n. Let $(X, R) : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ be *i.i.d.* , then the **reward process** $Q : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ is defined as the accumulated reward earned by time t as $Q_t \triangleq \sum_{i=1}^{N_t} R_i$.



Example 1.2. Consider a renewal sequence $S : \Omega \to \mathbb{R}+^{\mathbb{N}}$ with *i.i.d.* inter-renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. Consider an *i.i.d.* renewal sequence $R : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ defined as $R_n \triangleq 1$ for all $n \in \mathbb{N}$. Then the reward process $Q : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is the same as the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with the renewal sequence S.

Example 1.3. Consider a renewal sequence $S : \Omega \to \mathbb{R}+^{\mathbb{N}}$ with *i.i.d.* inter-renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. Consider an *i.i.d.* renewal sequence $R : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ defined as $R_n \triangleq X_n$ for all $n \in \mathbb{N}$. Then the reward process $Q : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is the last renewal instant $Q_t = S_{N_t}$ for all times $t \in \mathbb{R}_+$.

Theorem 1.4 (renewal reward). *Consider a renewal reward process $Q : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ with associated* i.i.d. *renewal reward sequence $(X, R) : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ where the mean of absolute value of reward $\mathbb{E}|R_1|$ and mean of absolute value of renewal duration $\mathbb{E}|X_1|$ are finite. Then the empirical average of reward converges, almost surely and in mean, i.e.*

$$\lim_{t\to\infty} \frac{Q_t}{t} = \frac{\mathbb{E}R_1}{\mathbb{E}X_1} \text{ a.s. }, \qquad \lim_{t\to\infty} \frac{\mathbb{E}Q_t}{t} = \frac{\mathbb{E}R_1}{\mathbb{E}X_1}.$$

Proof. We can write the rate of accumulated reward as $\frac{Q_t}{t} = \left(\frac{Q_t}{N_t}\right)\left(\frac{N_t}{t}\right)$. From the strong law of large numbers we obtain that, $\lim_{t\to\infty} \frac{1}{N_t} \sum_{i=1}^{N_t} R_i = \mathbb{E}R_1$, and from the strong law for counting processes we have $\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}X_1}$.

Since $N_t + 1$ is a stopping time for the renewal reward sequence $((X_1, R_1), (X_2, R_2), \dots)$, it follows from Wald's lemma,

$$\mathbb{E}Q_t = \mathbb{E}\left[\sum_{i=1}^{N_t} R_i\right] = \mathbb{E}\left[\sum_{i=1}^{N_t+1} R_i\right] - \mathbb{E}R_{N_t+1} = (m_t + 1)\mathbb{E}R_1 - \mathbb{E}R_{N_t+1}.$$

Defining $g(t) \triangleq \mathbb{E}R_{N_t+1}$, using elementary renewal theorem, it suffices to show that $\lim_{t\to\infty} \frac{g(t)}{t} = 0$. Observe that R_{N_t+1} is a regenerative process with the regenerative sequence being the renewal instants S, since the nth segment is $\xi_n \triangleq (X_n, R_n)$ and the sequence (X, R) is *i.i.d.* . Defining kernel function $K(t) \triangleq \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1 > t\}}]$, we can write the renewal function for $g(t)$ as

$$g(t) = \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1>t\}}] + \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1\leqslant t\}}] = K(t) + \int_0^t g(t-u)dF(u).$$

We observe that the kernel function $K : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded above as

$$K(t) \triangleq \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1>t\}}] = \int_t^\infty \mathbb{E}[R_1 \mid \{X_1 = u\}]dF(u) \leqslant \int_t^\infty \mathbb{E}[|R_1| \mid \{X_1 = u\}]dF(u).$$

Using the solution to renewal function, we can write $g = (1 + m) * K$ in terms of renewal function m and kernel function K. From finiteness of $\mathbb{E}|R|$, it follows that $\lim_{t\to\infty} K(t) = 0$, and we can choose T such that $|K(u)| \leqslant \epsilon$ for all $u \geqslant T$. Hence, for all $t \geqslant T$, we have

$$\begin{aligned}\frac{|g(t)|}{t} &\leqslant \frac{|K(t)|}{t} + \int_0^{t-T} \frac{|K(t-u)|}{t}dm(u) + \int_{t-T}^t \frac{|K(t-u)|}{t}dm(u)\\ &\leqslant \frac{\epsilon}{t} + \frac{\epsilon m(t-T)}{t} + \mathbb{E}\,|R_1|\,\frac{(m(t) - m(t-T))}{t}.\end{aligned}$$

Taking limits and applying elementary renewal and Blackwell's theorem, we get $\limsup_{t\to\infty} \frac{|g(t)|}{t} \leqslant \frac{\epsilon}{\mathbb{E}X_1}$. The result follows since $\epsilon > 0$ was arbitrary. □

Lemma 1.5 (Inspection Paradox). *For a renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with inter-arrival times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$,*

$$\mathbb{E}[X_{N_t+1}] \geqslant \mathbb{E}[X_1]$$

Proof. It suffices to show that $g(t) \triangleq P\{X_{N_t+1} > x\} \geqslant \bar{F}(x)$ for all $x, t \in \mathbb{R}_+$. To this end, we first observe that the segment of X_{N_t+1} during the nth renewal period $[S_{n-1}, S_n)$ is $\xi_n = (X_n, X_n)$. It follows that, X_{N_t+1} is a regenerative process with regeneration instants being the renewal sequence S. We can write the renewal equation for g as

$$g(t) = P\{X_{N_t+1} > x, S_1 > t\} + P\{X_{N_t+1} > x, S_1 \leqslant t\} = \bar{F}(x \vee t) + \mathbb{E}[g(t - S_1)\mathbb{1}_{\{S_1 \leqslant t\}}].$$

Defining kernel function $K : \mathbb{R}_+ \to [0,1]$ by $K(t) \triangleq P\{X_{N_t+1} > x, S_1 > t\} = \bar{F}(x \vee t)$, the solution to the renewal equation is given by $g = K * (1 + m)$. That is, we can write

$$g(t) = \begin{cases} \bar{F}(x)(1 + m(t)), & t \leqslant x,\\ 1 + \int_{t-x}^t (\bar{F}(x) - \bar{F}(t-u))dm(u), & x < t. \end{cases}$$

□

Remark 1. R_{N_t+1} is possibly dependent on X_{N_t+1}. Hence, due to the inspection paradox, R_{N_t+1} is may have a different distribution to R_1.

Need to revisit this

Lemma 1.6. *Reward $R_{N(t)+1}$ at the next renewal has different distribution than R_1.*

Proof. Notice that $R_{N(t)+1}$ is related to $X_{N(t)+1}$ which is the length of the renewal interval containing the point t. We have seen that larger renewal intervals have a greater chance of containing t. That is, $X_{N(t)+1}$ tends to be larger than a ordinary renewal interval. Since $R_{N(t)+1}$ is a regenerative process, we can formally write its tail probability as

$$f(t) = P\{R_{N(t)+1} > x\} = K(t) + (m * K)(t),$$

where in terms of the distribution functions F, H for inter-renewal times and rewards we can write the the kernel

$$K(t) = P\{R_{N(t)+1} > x, X_1 > t\} = P\{R_1 > x, X_1 > t\} \leqslant \bar{F}(t).$$

It follows that $f(t) \leqslant \bar{F}(t) + (m * \bar{F})(t) = 1$. □

Lemma 1.7. ¡¡¡¡¡¡¡ *HEAD Renewal reward theorem applies to a reward process $R(t)$ that accrues reward continuously over a renewal duration. The total reward in a renewal duration X_n remains R_n as before, with the sequence$((X_n, R_n) : n \in \mathbb{N})$ being* i.i.d. . *======= Renewal reward theorem applies to a reward process $R(t)$ that accrues positive reward continuously over a renewal duration. The total reward in a renewal duration X_n remains R_n as before, with the sequence$((X_n, R_n) : n \in \mathbb{N})$ being iid. ¿¿¿¿¿¿¿ 7516c998a3884b2c273f6d16145aac03f7af4963*

Proof. Let the process $R(t)$ denote the accumulated reward till time t, when the reward accrual is continuous in time. Now, defining $R_n = R(S_n) - R(S_{n-1}) > 0$, it follows that

$$\frac{\sum_{n=1}^{N(t)} R_n}{t} \leq \frac{R(t)}{t} \leq \frac{\sum_{n=1}^{N(t)+1} R_n}{t}.$$

Result follows from application of strong law of large numbers. □

1.1 Limiting empirical average of age and excess times

To determine the average value of the age of a renewal process, consider the following gradual reward process. We assume the reward rate to be equal to the age of the process at any time t, and

$$R(t) = \int_0^t A(u)du.$$

Observe that age is a linear increasing function of time in any renewal duration. In nth renewal duration, it increases from 0 to X_n, and the total reward $R_n = X_n^2/2$. Hence, we obtain from the renewal reward theorem

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t A(u)du = \frac{\mathbb{E}R_n}{\mathbb{E}X_n} = \frac{\mathbb{E}X^2}{2\mathbb{E}X}.$$

Example 1.8. Since the accumulated excess time during one renewal cycle is $\int_0^{X_n}(X_n - t)dt$, the limiting empirical average of excess time $Y(t) = t - S_{N(t)}$ can be found using the renewal reward theorem is

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Y(u)du = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}.$$

Example 1.9. The limiting average of current renewal interval $X_{N(t)} = A(t) + Y(t) = S_{N(t)+1} - S_{N(t)}$ can be computed directly as the sum of two limiting averages, or from the application of renewal reward theorem with accrued reward in one renewal interval being $\int_0^{X_n} X_n dt = X_n^2$, to get

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t X_{N(u)+1}du = \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]}.$$

We see that this limit is always greater than $\mathbb{E}[X]$, except when X is constant. Such a result was to be expected in view of the inspection paradox, since we can show that $\lim_{t\to\infty} \mathbb{E}[X_{N(t)+1}] = \lim_{t\to\infty} \frac{1}{t} \int_0^t X_{N(u)+1}du$.

Example 1.10. It can be shown, under certain regularity conditions, that

$$\lim_{t\to\infty} \mathbb{E}R_{N(t)+1} = \lim_{t\to\infty} \frac{1}{t} \int_0^t R_{N(u)+1}du = \frac{\mathbb{E}[R_1 X_1]}{\mathbb{E}[X_1]}.$$

If reward is a monotonically increasing function of renewal interval, then we get that $\lim_{t\to\infty} \mathbb{E}R_{N(t)+1} \geqslant \mathbb{E}R_1$ from Chebyshev's inequality.

1.2 Stationary probability and empirical average

Theorem 1.11. *For an alternative renewal process $W = \{W(t) \in \{0,1\} : t \geqslant 0\}$ the stationary probability of being on is same as the limiting average time spent in the on duration if the renewal duration has finite mean. That is,*

$$\lim_{t\to\infty} P\{W(t) = 1\} = \lim_{t\to\infty} \frac{1}{t}\int_0^t W(u)du.$$

Proof. Suppose for an alternating renewal process, we earn at a unit rate in on state. The aggregate reward in one renewal duration X_n is the on time Z_n in that duration.

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t W(u)du = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}Z_n}{\mathbb{E}X_n} = \lim_{t\to\infty} P(\text{on at time } t).$$

□

1.3 Patterns

Let $X : \Omega \to \{0,1\}^{\mathbb{N}}$ be a *i.i.d.* Bernoulli sequence with $\mathbb{E}[X_1] = p$. Let $x = (x_1, \cdots, x_m)$ be a pattern and define the first hitting time of the pattern x as $S_x \triangleq \inf\{n \in \mathbb{N} : X_n = x_m, X_{n-1} = x_{m-1}, \cdots, X_{n-m+1} = x_1\}$. So, one can see that $(S_x^n, n \in \mathbb{N})$ is a delayed renewal sequence with *i.i.d.* inter-arrival times $T_x^k \triangleq S_x^k - S_x^{k-1}, k \geqslant 2$.

Example 1.12. Let us say that the pattern is (1). So, $P(S_1 = k) = (1-p)^{k-1}p$ and $\mathbb{E}[S_1] = 1/p$.

Example 1.13. Let us say that the pattern is $(0,1)$. So,

$$\begin{aligned}
\mathbb{E}[S_x] &= \mathbb{E}[S_x \mathbb{1}_{\{X_1=0\}}] + \mathbb{E}[S_x \mathbb{1}_{\{X_1=1\}}] \\
&= \mathbb{E}[S_x \mathbb{1}_{\{X_1=0,X_2=1\}}] + \mathbb{E}[S_x \mathbb{1}_{\{X_1=0,X_2=0\}}] + \mathbb{E}[S_x \mathbb{1}_{\{X_1=1\}}] \\
&= \mathbb{E}[(1+S_x')\mathbb{1}_{\{X_1=1\}}] + \mathbb{E}[S_x \mathbb{1}_{\{X_1=0,X_2=1\}}] + \mathbb{E}[S_x \mathbb{1}_{\{X_1=0,X_2=0\}}] \\
&= p\mathbb{E}[S_x] + p + 2p\bar{p} + \bar{p}\mathbb{E}[(1+S_x)\mathbb{1}_{\{X_1=0\}}].
\end{aligned}$$

See that

$$\begin{aligned}
\mathbb{E}[S_x \mathbb{1}_{\{X_1=0\}}] &= 2p\bar{p} + \bar{p}\mathbb{E}[(1+S_x)\mathbb{1}_{\{X_1=0\}}] \\
&= 2p\bar{p} + \bar{p}^2 + \bar{p}\mathbb{E}[S_x \mathbb{1}_{\{X_1=0\}}] \\
\mathbb{E}[S_x \mathbb{1}_{\{X_1=0\}}] &= \frac{2p\bar{p} + \bar{p}^2}{1-\bar{p}}.
\end{aligned}$$

Hence, $\mathbb{E}[S_x] = 1/p\bar{p}$.

2 Patterns

Let $(X_n \in \{0,1\} : n \in \mathbb{N})$ be an *iid* Bernoulli sequence with common mean $\mathbb{E}X_n = p$. Then, we are interested in finding the mean time to find first occurrence of a string pattern. We define the first hitting time to a pattern $x = (x_1, \ldots, x_p)$ as

$$S_x \triangleq \inf\{n \in \mathbb{N} : X_n = x_p, \ldots, X_{n-p+1} = x_1\}.$$

Let S_x^k be the kth time the process X hits patten x, where

$$S_x^k \triangleq \inf\{n > S_x^{k-1} : X_n = x_p, \ldots, X_{n-p+1} = x_1\}.$$

It follows that $(S_x^k : k \in \mathbb{N})$ forms a delayed renewal sequence, where $T_x(k) \triangleq S_x^k - S_x^{k-1}$ are *iid* for $k \geqslant 2$.

2.1 Hitting time to pattern (1)

First we consider the simplest example when $x = (1)$. One way to solve this problem is to consider S_x as a random variable and find its distribution. For example, when $x = 1$ we can write

$$P\{S_1 = k\} = \bar{p}^{k-1}p.$$

We observe that S_1 is a geometric random variable of the time to first success, with its mean as the reciprocal of *iid* failure probability $\bar{p}$. Second way to solve this is via renewal function approach. We can write

$$\mathbb{E}S_1 = \mathbb{E}S_1 1_{\{X_1=0\}} + \mathbb{E}S_1 1_{\{X_1=1\}} = \bar{p} + p\mathbb{E}(1+S_1) = 1 + p\mathbb{E}S_1.$$

2.2 Hitting time to pattern $(0,1)$

We consider the two length pattern $x = (0,1)$, then $S_x = \inf\{n \in \mathbb{N} : X_n = 1, X_{n-1} = 0\}$. We can again model this hitting time as a random variable, however finding its distribution is slightly more complicated. We next attempt the renewal function approach, to get

$$\mathbb{E}S_x = \mathbb{E}S_x 1_{\{X_1=0\}} + \mathbb{E}S_x 1_{\{X_1=1\}} = \mathbb{E}S_x 1_{\{X_2=1,X_0=0\}} + \mathbb{E}S_x 1_{\{X_2=0,X_1=0\}} + p\mathbb{E}(1+S_x).$$

We recognize that the second term on the right hand side can be written as

$$\mathbb{E}S_x 1_{\{X_2=0,X_1=0\}} = \bar{p}\mathbb{E}(1+S_x)1_{\{X_1=0\}} = \bar{p}^2 + \bar{p}\mathbb{E}S_x 1_{\{X_1=0\}} = \bar{p}^2 + \bar{p}\mathbb{E}S_x - \bar{p}p\mathbb{E}(1+S_x).$$

Combining the above two results, we can write

$$\mathbb{E}S_x = 2p\bar{p} + \bar{p}^2 + \bar{p}\mathbb{E}S_x + p^2\mathbb{E}(1+S_x) = 1 + (\bar{p}+p^2)\mathbb{E}S_x.$$

2.3 Hitting time to pattern x

A general approach is to model $X_n^p = (X_n, X_{n-1}, X_{n-p+1})$ as a p-dimensional Markov chain, and find the hitting time to state x of the joint process $X^p = (X_n^p : n \in \mathbb{N})$. We observe that successive times to hit a pattern x is a delayed renewal process in general. If the time to hit pattern x is same as time duration between two successive hits to pattern x, these instants form a renewal process. From the renewal reward process, we can write the mean inter-renewal duration as the number of hits to pattern x, as

$$\mathbb{E}(S_x^k - S_x^{k-1}) = \lim_{n\to\infty}\frac{1}{N}\sum_{n=1}^{N} 1_{\{X_n^p = x\}} = \lim_{n\to\infty}\frac{1}{N}\sum_{n=1}^{N} 1_{\{X_n = x_p,\dots,X_{n-p+1}=x_1\}} = P\{X_n = x_p,\dots,X_{n-p+1} = x_1\}.$$

Mean hitting time to pattern x is equal to mean hitting time to a sub-pattern, and then hitting time from the sub-pattern to the pattern x. That is, in general we can write

$$\mathbb{E}T_x = \mathbb{E}T_x$$

A Chebyshev's sum inequality

Theorem A.1. *Consider two non-decreasing positive measurable functions $f, g : \mathbb{R} \to \mathbb{R}_+$ and a random variable $X : \Omega \to \mathbb{R}$. Then, $\mathbb{E}f(X)g(X) \geqslant \mathbb{E}f(X)\mathbb{E}g(X)$.*

Proof. Consider a random sequence $Y : \Omega \to \mathbb{R}^2$ to be *i.i.d.* replicas of $X : \Omega \to \mathbb{R}$ and the product $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))$. For non-decreasing functions f, g, we have

$$(f(Y_1)-f(Y_2))(g(Y_1)-g(Y_2)) = (f(Y_1)-f(Y_2))(g(Y_1)-g(Y_2))\mathbb{1}_{\{Y_1 \geqslant Y_2\}} + (f(Y_1)-f(Y_2))(g(Y_1)-g(Y_2))\mathbb{1}_{\{Y_1 < Y_2\}}.$$

Defining $m_f \triangleq \mathbb{E}f(X)$ and $m_g \triangleq \mathbb{E}g(X)$, we observe that since f, g are increasing, so are $f - m_f$ and $g - m_g$. Let $x_f \triangleq \inf\left\{x \in \mathbb{R} : f(x) - m_f \geqslant 0\right\}$ and $x_g \triangleq \inf\left\{x \in \mathbb{R} : g(x) - m_g \geqslant 0\right\}$. Then, for $x < x_f \wedge x_g$ and $x > x_f \vee x_g$ we have $(f - m_f)(g - m_g) \geqslant 0$ and □