Lecture-18: Embedded Markov Chain and Holding Times

1 State Evolution

For a homogeneous Markov process $X : \Omega \to \mathcal{X}^{\mathbb{R}}_+$ on countable state space $\mathcal{X} \subseteq \mathbb{R}$ that has right continuous sample paths with left limits (rcll), we wish to characterize the transition kernel $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X}\times\mathcal{X}}$. In particular, we wish to know following probabilities

$$P_{xy}(0,t) = P(\{X_{s+t} = y\} \mid \{X_s = x\}), \quad t \geqslant 0.$$

To this end, we define the sojourn time in any state, the jump times, and the jump transition probabilities.

1.1 Jump and sojourn times

Definition 1.1. We denote the natural filtration for the stochastic process X by $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in \mathbb{R}_+)$ where the history until any time $t \in \mathbb{R}_+$ is $\mathcal{F}_t = \sigma(X_u, u \leqslant t)$.

Definition 1.2. The **jump times** of a right continuous stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ are defined as

$$S_0 \triangleq 0, \qquad S_n \triangleq \inf\{t > S_{n-1} : X_t \neq X_{S_{n-1}}\}.$$

Lemma 1.3. *Jump times $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ are stopping times adapted to the natural filtration of the process X.*

Proof. It is clear that the event $\{S_n \leqslant t\}$ is completely determined by the history $\mathcal{F}_t$ until time t. □

Definition 1.4. We denote the state of the process at nth stopping time S_n as $Z_n \triangleq X_{S_n}$. The **sojourn time** of this process staying in state Z_{n-1} is $T_n \triangleq S_n - S_{n-1}$.

Definition 1.5. Let $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ be the counting process associated with jump times sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. That is, the number of jumps in $(0,t]$ is denoted by

$$N_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}.$$

Remark 1. From the definition of jump instants, it follows that the history until time t is

$$\mathcal{F}_t \subseteq \sigma((Z_0,S_0),(Z_1,T_1),\dots,(Z_{N_t},T_{N_t}),T_{N_t+1}).$$

We can verify that $\mathcal{F}_{S_n} = \sigma((Z_i,T_i),\ 0 \leqslant i \leqslant n)$.

Lemma 1.6. *For a homogeneous CTMC, each sojourn time $T_n : \Omega \to \mathbb{R}_+$ is a continuous memoryless random variable, and the sequence of sojourn times $(T_j : j \geqslant n)$ is independent of the past $\mathcal{F}_{S_{n-1}}$ conditioned on Z_{n-1}.*

Proof. We observe that the sojourn time T_n equals the excess time $Y_{S_{n-1}}$ in state Z_{n-1} starting at time S_{n-1}. Using the strong Markov property, we can write the conditional complementary distribution of T_n given any historical event $H \in \mathcal{F}_{S_{n-1}}$ and $y \geqslant 0$ as

$$P(\{T_n > u\} \mid \{Z_{n-1} = x\} \cap H) = P(\{Y_{S_{n-1}} > u\} \mid \{X_{S_{n-1}} = x\} \cap H) = \exp(-u\nu_x) = \bar{F}_x(u).$$

□

Corollary 1.7. *If $X_n = x$, then the random variable T_{n+1} has an exponential random distribution with rate ν_x.*

Remark 2. Inverse of mean sojourn time in state x is called the **transition rate** out of state x denote by $\nu_x = (\mathbb{E}_x T_1)^{-1}$. Recall that a state x is instantaneous if $\nu_x = \infty$, stable if $0 < \nu_x < \infty$, and absorbing if $\nu_x = 0$.

Definition 1.8. A pure jump CTMC with

(i) all stable states and $\inf_{x \in \mathcal{X}} \nu_x \geqslant \nu > 0$ is called **stable**, and

(ii) $\sup_{x \in \mathcal{X}} \nu_x \leqslant \nu < \infty$ is called **regular**.

Proposition 1.9. *For a stable CTMC, the jump times are almost surely finite stopping times.*

Proof. We observe that the jump times are sum of independent exponential random variables. Further by coupling, we can have a sequence of *i.i.d.* random variables $\overline{T} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, such that $T_n \leqslant \overline{T}_n$ almost surely and $\mathbb{E}\overline{T}_n = \frac{1}{\nu}$ for each $n \in \mathbb{N}$. Hence, we have

$$S_n = \sum_{i=1}^{n} T_i \leqslant \sum_{i=1}^{n} \overline{T}_i \triangleq \overline{S}_n.$$

Since $\overline{S}_n$ is the nth arrival instant of a Poisson process with finite rate ν, it follows that $\overline{S}_n$ and hence S_n is finite almost surely. $\square$

Make the following theorem iff

Proposition 1.10. *For a regular CTMC, $P\{N_t < \infty\} = 1$ for all finite times $t \in \mathbb{R}_+$.*

Proof. We observe that the jump times are sum of independent exponential random variables. Further by coupling, we can have a sequence of *i.i.d.* random variables $\underline{T} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, such that $T_n \geqslant \underline{T}_n$ almost surely and $\mathbb{E}\underline{T}_n = \frac{1}{\nu}$ for each $n \in \mathbb{N}$. Hence, we have for each $n \in \mathbb{N}$,

$$S_n = \sum_{i=1}^{n} T_i \geqslant \sum_{i=1}^{n} \underline{T}_i \triangleq \underline{S}_n.$$

Let $\underline{m}(t)$ be the associated renewal function with the sequence $\underline{T}$, then

$$m(t) = \sum_{n \in \mathbb{N}} P\{S_n \leqslant t\} \leqslant \sum_{n \in \mathbb{N}} P\{\underline{S}_n \leqslant t\} = \underline{m}(t).$$

Since the inter-renewal times have finite means, it follows that $\underline{m}(t)$ and hence $m(t)$ is finite. This implies that positive random variable N_t has finite mean, and hence is finite almost surely for any finite $t \in \mathbb{R}_+$. $\square$

1.2 Jump process

Definition 1.11. The **jump process** is a discrete time process $Z : \Omega \to \mathcal{X}^{\mathbb{N}}$ derived from the continuous time stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ by sampling at jump times. This is also sometimes referred to as the **embedded DTMC** of the pure jump CTMC X.

Definition 1.12. The corresponding **jump transition probabilities** are defined

$$p_{xy} \triangleq P_{xy}(S_{n-1}, S_n) = P(\{X_{S_n} = y\} \mid \{X_{S_{n-1}} = x\}), \quad x, y \in \mathcal{X}.$$

Remark 3. From the strong Markov property and the time-homogeneity of the CTMC X, we see that $P_{xy}(S_{n-1}, S_n) = P_{xy}(0, T_1)$.

Lemma 1.13. *For any right continuous with left limits stochastic process X, the sum of jump transition probabilities $\sum_{y \neq x} P_{xy}(S_{n-1}, S_n) = 1$ for all $X_{S_{n-1}} = x \in \mathcal{X}$.*

Proof. It follows from law of total probability. $\square$

Lemma 1.14. *For a homogeneous CTMC, the jump probability from state $X_{S_{n-1}}$ to state X_{S_n} depends solely on $X_{S_{n-1}}$ and is independent of jump instants.*

Proof. We can write the joint probability of state $Z_n = y$ and $T_n > u$ for any $y \in \mathcal{X}, u \in \mathbb{R}_+$ conditioned on the state $Z_{n-1} = x$ and any historical event $H \in \mathcal{F}_{S_{n-1}}$ for any states $x, y \in \mathcal{X}$, using the definition of excess time $Y_t = S_{N_t+1} - t$, the strong Markov property, the time-homogeneity of CTMC X, and memoryless property of excess time Y, as

$$P(\{T_n > u, Z_n = y\} \mid \{Z_{n-1} = x\} \cap H) = P(\{X_{u+Y_u} = y\} \mid \{X_u = x\}) P(\{Y_0 > u\} \mid \{X_0 = x\}) = P_{xy}(T_1)\bar{F}_x(u).$$

We define $p_{xy} \triangleq P_{xy}(T_1)$, and hence the result follows and we can write

$$P(\{T_n > u, Z_n = y\} \mid \{Z_{n-1} = x\}) = P(\{T_n > u\} \mid \{Z_{n-1} = x\}) P(\{Z_n = y\} \mid \{Z_{n-1} = x\}) = p_{xy} e^{-u\nu_x}.$$

This implies that sojourn times and jump instant probabilities are independent. $\square$

Maybe these can be merged with previous theorems

Corollary 1.15. *The matrix $P = (p_{xy} : x, y \in \mathcal{X})$ is stochastic, and if $\nu_x > 0$ then $p_{xx} = 0$.*

Proof. Recall $p_{xy} = P_{xy}(S_1)$. If $\nu_x > 0$, then $\lim_{u\to\infty} P(\{Y_0 > u\} \,|\, \{X_0 = x\}) = 0$, and hence S_1 is finite almost surely. By definition $X_{S_1} \neq X_0 = x$, and hence $p_{xx} = 0$. □

Remark 4. If $\nu_x = 0$, then for any $u \geqslant 0$, we have $P(\{Y_0 > u\} \,|\, \{X_0 = x\}) = 1$, and hence $S_1 = \infty$ almost surely whenever $X(0) = x$. By convention, we set $p_{xx} = 1$ and $p_{xy} = 0$ for all states $y \neq x$.

Theorem 1.16. *For a pure jump CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ on state space $\mathcal{X}$, if S_n is a proper stopping time for some $n \in \mathbb{N}$. Then for all states $x, y \in \mathcal{X}$ and duration $u \geqslant 0$, we have*

$$P(\{T_{n+1} > u, Z_{n+1} = y\} \,|\, \{X_0 = x_0, \dots, Z_n = x, S_0 \leqslant s_0, \dots, S_n \leqslant s_n\}) = p_{xy} e^{-u\nu_x}.$$

Proof. Since the history of the process until stopping time S_n is given by $\mathcal{F}_{S_n} = \sigma((Z_i, T_i) : 0 \leqslant i \leqslant n)$, and $H \cap \{Z_n = x\} = \{Z_0 = x_0, \dots, Z_n = x, S_0 \leqslant s_0, \dots, S_n \leqslant s_n\} \in \mathcal{F}_{S_n}$. Using strong Markov property and time-homogeneity of the CTMC X, we have

$$P(\{T_{n+1} > u, Z_{n+1} = y\} \,|H \cap \{Z_n = x\}) = P_x\{S_1 > u, Z_1 = x\}.$$

The result follows from the previous Lemma 1.14. □

Corollary 1.17. *For a time-homogeneous CTMC, the transition probabilities $(p_{xy} : x, y \in \mathcal{X})$ and sojourn times $T : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ are independent.*

Corollary 1.18. *The jump process is a homogeneous Markov chain with countable state space $\mathcal{X}$.*

Example 1.19 (Poisson process). For a Poisson counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with time-homogeneous rate λ, the countable state space is $\mathbb{Z}_+$, and transition rate $\nu_i = \lambda$ for each $i \in \mathbb{Z}_+$. It follows from the memoryless property of exponential random variables, that

$$P(\{Y_u > t\} \,|\, \{N_u = i\}) = P\{S_1 > t\} = e^{-\lambda t}.$$

Further, the embedded Markov chain or the jump process is given by the initial state $N_0 = 0$ and the transition probability matrix $P = (p_{ij} : i, j \in \mathbb{Z}_+)$ where $p_{i,i+1} = 1$ and $p_{ij} = 0$ for $j \neq i+1$. This follows from the definition of T_1, since $p_{ij} = P(\{N_{T_1} = j\} \,|\, \{N_0 = i\}) = \mathbb{1}_{\{j=i+1\}}$.

Theorem 1.20. *A pure-jump homogeneous CTMC whose embedded DTMC is recurrent is regular.*

Proof. From the recurrence of embedded chain, there exists a state $x \in \mathcal{X}$ with holding rate $\nu_x > 0$. Let $N_x(n) = \sum_{i=1}^{n} \mathbb{1}_{\{Z_i = x\}}$ be the number of visits to state x in first n transitions and T_i^x be the ith sojourn time in state x. From the recurrence of the embedded chain, this state x occurs infinitely often, i.e. $\lim_{n\in\mathbb{N}} N_x(n) = \infty$ almost surely. For each state $x \in \mathcal{X}$, the sojourn time sequence $T^x : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* and exponentially distributed with mean $\mathbb{E}T_i^x = \frac{1}{\nu_x} < \infty$. Since $S_n \geqslant \sum_{i=1}^{N_x(n)} T_i^x$, we get that

$$m(t) = \sum_{n\in\mathbb{N}} P\{S_n \leqslant t\} \leqslant \sum_{n\in\mathbb{N}} P\left\{\sum_{i=1}^{N_x(n)} T_i^x \leqslant t\right\} = \nu_x t.$$

It follows that N_t is almost surely finite for any finite time $t \in \mathbb{R}_+$. □

1.3 Alternative construction of CTMC

Definition 1.21. Let $Z : \Omega \to \mathcal{X}^{\mathbb{N}}$ be a discrete time Markov chain with a countable state space $\mathcal{X} \subseteq \mathbb{R}$, and the transition probability matrix $P : \mathcal{X} \times \mathcal{X} \to [0,1]$ a stochastic matrix. Further, we let $\nu : \mathcal{X} \to \mathbb{R}_+$ be the set of transition rates such that $p_{xx} = 0$ if $\nu_x > 0$. For any initial state $Z_0 \in \mathcal{X}$, defining we can define a right continuous with left limits piece-wise constant stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ inductively as

$$X_t \triangleq Z_{n-1}, \quad t \in [S_{n-1}, S_n),$$

where $S_0 = 0$ and nth transition instant $S_n \triangleq \sum_{i=1}^{n} T_i$, where T_n is a random variable independent of (Z_0, T_0), …, (Z_{n-2}, T_{n-2}), T_{n-1} and distributed exponentially with rate ν_x if $Z_{n-1} = x$. We define the the number of transitions until time t by $N_t = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$.

Remark 5. From the definition, any sample path of the process is right-continuous with left limits, and has countable state space $\mathcal{X}$.

Remark 6. We observe that the history of the process until time t is given by $\sigma((Z_0,T_1),\dots,(Z_{N_t},T_{N_t+1}))$.

Remark 7. A necessary condition for the process X to be defined on index set $\mathbb{R}_+$, is that for each $t \in \mathbb{R}_+$, there exists an n such that $S_n \leqslant t < S_{n+1}$. That is, $P\{N_t < \infty\} = P\{S_\infty > t\} = 1$ for all $t \in \mathbb{R}_+$. This is equivalent to $P\{S_\infty = \infty\} = 1$, or $P\{S_\infty < \infty\} = 0$. Let $\omega \in \{S_\infty < \infty\}$, then we can't define the process for $t > S_\infty$.

Remark 8. We will show that X is a CTMC. Recall that, a regular CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ has $P\{N_t < \infty\} = 1$ for all $t \in \mathbb{R}_+$.

Example 1.22 (Non-regular CTMC). For the countable state space $\mathcal{X}$, consider the probability transition matrix P such that $p_{i,i+1} = 1$ and the exponential holding times with rate $\nu_i = i^2$ for each state $i \in \mathbb{N}$. Clearly, $\sup_{i\in\mathbb{N}} \nu_i = \infty$, and hence it is not regular.

Lemma 1.23. *Conditioned on the process state at the beginning of an interval, the increments of the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is independent of the past, and depends only on the duration of the increment. That is, for a historical event $H \in \mathcal{F}_s$ and state $x \in \mathcal{X}$,*

$$P(\{N_t - N_s = k\} \mid \{X_s = x\} \cap H) = P(\{N_{t-s} = k\} \mid \{X_0 = x\}).$$

Proof. From the independence of inter-transition times, we know that T_{N_s+j} is independent of history $\mathcal{F}_s$ for $j \geqslant 2$ conditioned on the process state $X_s = x$. Further, from the memoryless property of an exponential random variable, the excess time Y_s is independent of the age $A_s = s - S_{N_s}$, and is identically distributed to T_{N_s+1}. Therefore, the the conditional distribution of $(Y_s, T_{N_s+2}, \dots, T_{N_s+k})$ given the current process state $X_s = x$, is identical to that of the conditional distribution of inter-transition times $(T_1, T_2, \dots, T_k)$ given initial state $X_0 = x$. Hence for any historical event $H \in \mathcal{F}_s$ and state $x \in \mathcal{X}$, we can write the conditional probability of increment $N_t - N_s$ for $t > s$, as

$$P(\{N_t - N_s = k\} \mid \{X_s = x\} \cap H) = P(\left\{ Y_s + \sum_{i=N_s+2}^{N_s+k} T_i \leqslant t - s < Y_s + \sum_{i=N_s+2}^{N_s+k+1} T_i \right\} \mid \{X_s = x\} \cap H) = P_x\{N_{t-s} = k\}.$$

□

Proposition 1.24. *The stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ constructed in Definition 1.21 is a time-homogeneous CTMC.*

Proof. For states $x, y \in \mathcal{X}$, we can write the probability of process being in state y, conditioned on any historical events $H \in \mathcal{F}_s$ as

$$P(\{X_t = y\} \mid \{X_s = x\} \cap H) = \sum_{k\in\mathbb{Z}_+} P(\{X_t = y, N_t - N_s = k\} \mid \{X_s = x\} \cap H).$$

From the construction of process X in Definition 1.21, conditional independence of counting process and time homogeneity from Lemma 1.23, and definition of conditional probability, we can write for each $k \in \mathbb{N}$,

$$\begin{aligned} P(\{X_t = y, N_t - N_s = k\} \mid \{X_s = x\} \cap H) &= P(\left(\cap_{i\in[k-1]} \{Z_i \in \mathcal{X} \setminus \{Z_{i-1}\}\}\right) \cap \{Z_k = y\} \mid \{Z_0 = x\}) P_x\{N_{t-s} = k\} \\ &= P_x\{X_{t-s} = y, N_{t-s} = k\}. \end{aligned}$$

Thus, we have shown the time homogeneity and Markov property for the process X. □

Theorem 1.25. *A stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ defined on countable state space $\mathcal{X} \subseteq \mathbb{R}$ and having right continuous sample paths with left limits, is a CTMC iff*

i_ sojourn times are independent and exponentially distributed with rate ν_x where $X_{S_{n-1}} = x$, and

ii_ jump transition probabilities $p_{xy} = P_{xy}(S_{n-1}, S_n)$ are independent of jump times S_n, such that $\sum_{y\neq x} p_{xy} = 1$.