Lecture-24: Martingales

1 Martingales

Definition 1.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A **filtration** is an increasing sequence of σ-fields denoted by $\mathcal{F}_\bullet = (\mathcal{F}_n \subseteq \mathcal{F} : n \in \mathbb{N})$, with nth σ-field denoted by $\mathcal{F}_n$.

Definition 1.2. For a discrete stochastic process $X : \Omega \to \mathbb{R}^{\mathbb{N}}$, its **natural filtration** is defined as

$$\mathcal{F}_n \triangleq \sigma(X_1, \dots, X_n).$$

Definition 1.3. A random sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ of random variables is said to be **adapted** to the filtration $\mathcal{F}_\bullet$ if $\sigma(X_n) \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$.

Remark 1. For any random sequence X adapted to a filtration $\mathcal{F}_\bullet$, we also have $\sigma(X_1, \dots, X_n) \subseteq \mathcal{F}_n$ for each $n \in \mathbb{N}$.

Definition 1.4. A discrete stochastic process $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ is said to be a **martingale** with respect to the filtration $\mathcal{F}_\bullet$ if for each $n \in \mathbb{N}$,

i_ **integrability.** $\mathbb{E}|X_n| < \infty$,

ii_ **adaptability.** $\sigma(X_n) \subseteq \mathcal{F}_n$,

iii_ **unbiasedness.** $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$.

If the equality in third condition is replaced by $\leqslant$ or $\geqslant$, then the process is called **supermartingale** or **submartingale**, respectively.

Corollary 1.5. *For a martingale X adapted to a filtration $\mathcal{F}_\bullet$, we have*

$$\mathbb{E}X_n = \mathbb{E}X_1, \qquad n \in \mathbb{N}.$$

Example 1.6 (Simple random walk). Let $\xi : \Omega \to \mathbb{R}^{\mathbb{N}}$ be a sequence of independent random variables with mean $\mathbb{E}\xi_i = 0$ and $\mathbb{E}|\xi_i| < \infty$ for each $i \in \mathbb{N}$. Let $\mathcal{F}_\bullet$ be the natural filtration of random sequence ξ, such that $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$ for each $n \in \mathbb{N}$. Consider the random sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ such that $X_n \triangleq \sum_{i=1}^n \xi_i$ for each $n \in \mathbb{N}$.

Then, the random sequence X is a martingale with respect to filtration $\mathcal{F}_\bullet$. This follows, since $\mathbb{E}X_n = 0$, and from the linearity of expectation and the finiteness of finitely many individual terms, the absolute sum $\mathbb{E}|X_n| \leqslant \sum_{i=1}^n \mathbb{E}|\xi|_i < \infty$. Further, we have

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n + \xi_{n+1}|\mathcal{F}_n] = X_n.$$

Example 1.7 (Product martingale). Let $\xi : \Omega \to \mathbb{R}^{\mathbb{N}}$ be a sequence of independent random variables with mean $\mathbb{E}\xi_i = 1$ and $\mathbb{E}|\xi_i| < \infty$ for each $i \in \mathbb{N}$. Let $\mathcal{F}_\bullet$ be the natural filtration of random sequence ξ, such that $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$ for each $n \in \mathbb{N}$. Consider the random sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ such that $X_n \triangleq \prod_{i=1}^n \xi_i$ for each $n \in \mathbb{N}$, then X is adapted to $\mathcal{F}_\bullet$.

Then, the random sequence X is a martingale with respect to filtration $\mathcal{F}_\bullet$. This follows, since $\mathbb{E}X_n = 1$, and from the independence and finiteness of finitely many individual terms the absolute product $\mathbb{E}|X_n| = \prod_{i=1}^n \mathbb{E}\xi_i < \infty$. Further, we have

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n \xi_{n+1}|\mathcal{F}_n] = X_n.$$

Example 1.8 (Branching process). Consider a population where each individual i can produce an independent random number of offsprings Z_i in its lifetime, given by a common distribution $P : \mathbb{Z}_+ \to [0,1]$ and finite mean $\mu = \sum_{j \in \mathbb{N}} jP_j < \infty$. Let X_n denote the size of the nth generation, which is same as the number of offsprings generated by $(n-1)$th generation. The discrete stochastic process $X : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ is called a branching process. Let $X_0 = 1$ and consider the natural filtration $\mathcal{F}_\bullet$ of X such that $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Then,

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i.$$

Conditioning on X_{n-1} yields, $\mathbb{E}[X_n|\mathcal{F}_{n-1}] = \mathbb{E}[\sum_{i=1}^{X_{n-1}} Z_i|\mathcal{F}_{n-1}] = \mathbb{E}[\sum_{i\in\mathbb{N}} Z_i \mathbb{1}_{\{i \leqslant X_{n-1}\}}|\mathcal{F}_{n-1}] = \sum_{i\in\mathbb{N}} \mathbb{E}[Z_i|\mathcal{F}_{n-1}]\mathbb{1}_{\{i\leqslant X_{n-1}\}} = \sum_{i=1}^{X_{n-1}} \mu = \mu X_{n-1}$. Applying expectation on both sides, and by induction on n, we get $\mathbb{E}[X_n] = \mu^n$. Consider a positive random sequence $Y : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ defined by $Y_n \triangleq \frac{X_n}{\mu^n}$ for each $n \in \mathbb{N}$. Then Y is a martingale with respect to filtration $\mathcal{F}_\bullet$ because $\mathbb{E}[Y_n] = 1$, the expectation of absolute value $\mathbb{E}|Y_n| = \frac{\mathbb{E}|X_n|}{\mu^n} = \frac{\mathbb{E}X_n}{\mu^n} = 1$, and

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = \frac{1}{\mu^{n+1}}\mathbb{E}[\sum_{i=1}^{X_n} Z_i|\mathcal{F}_n] = \frac{X_n}{\mu^n} = Y_n.$$

Example 1.9 (Doob's Martingale). Consider an arbitrary random sequence $Y : \Omega \to \mathbb{R}^{\mathbb{N}}$ with associated natural filtration $\mathcal{F}_\bullet$, and an arbitrary random variable $Z : \Omega \to \mathbb{R}$ such that$\mathbb{E}|Z| < \infty$. Then, a random sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined by $X_n \triangleq \mathbb{E}[Z|\mathcal{F}_n]$ for each $n \in \mathbb{N}$, is a martingale. The integrability condition can be directly verified, the sequence X is adapted to $\mathcal{F}_\bullet$ by definition of conditional expectation, and by the tower property of conditional expectation

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Z|\mathcal{F}_{n+1}]|\mathcal{F}_n] = \mathbb{E}[Z|\mathcal{F}_n] = X_n.$$

Example 1.10 (Centralized Doob sequence). For any sequence of random variables $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ with $\mathbb{E}|X_n| < \infty$ for all $n \in \mathbb{N}$ and its natural filtration $\mathcal{F}_\bullet$, the random variable $X_i - \mathbb{E}[X_i|\mathcal{F}_{i-1}]$ is zero mean for each $i \in \mathbb{N}$. Hence, the centralized zero mean sequence $Z : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined by $Z_n \triangleq \sum_{i=1}^n (X_i - \mathbb{E}[X_i|\mathcal{F}_{i-1}])$ for each $n \in \mathbb{N}$. Then the random sequence Z is adapted to the filtration $\mathcal{F}_\bullet$ and is a martingale with respect to this filtration $\mathcal{F}_\bullet$, provided $\mathbb{E}|Z_n| < \infty$. This is true because $\mathbb{E}|Z_n| \leqslant \sum_{i=1}^n \mathbb{E}|X_i - \mathbb{E}[X_i|\mathcal{F}_{i-1}]| \leqslant \sum_{i=1}^n \mathbb{E}|X_i| + \mathbb{E}[|X_i|\,|\mathcal{F}_{i-1}] < \infty$. Further, from the linearity and the tower property of conditional expectation, we have

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = \mathbb{E}[Z_n + X_{n+1} - \mathbb{E}[X_{n+1}|\mathcal{F}_n]|\mathcal{F}_n] = Z_n + \mathbb{E}[X_{n+1}|\mathcal{F}_n] - \mathbb{E}[X_{n+1}|\mathcal{F}_n] = Z_n.$$

Lemma 1.11. *Consider a filtration $\mathcal{F}_\bullet = (\mathcal{F}_n \subseteq \mathcal{F} : n \in \mathbb{N})$ on the probability space $(\Omega, \mathcal{F}, P)$. Consider a random sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ which is a martingale with respect to the filtration $\mathcal{F}_\bullet$, and a convex function $f : \mathbb{R} \to \mathbb{R}$. Then, the random sequence $Y : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined by $Y_n \triangleq f(X_n)$ for each $n \in \mathbb{N}$, is a submartingale with respect to the filtration $\mathcal{F}_\bullet$.*

Proof. The result is a direct consequence of Jensen's inequality for conditional expectations, since

$$\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] \geqslant f(\mathbb{E}[X_{n+1}|\mathcal{F}_n]) = f(X_n).$$

Check if this is true:

$$\mathbb{E}|Y_n| = \mathbb{E}|f(X_n)| \leqslant |\mathbb{E}[f(X_n)]|$$

If this is not true, what condition is needed for integrability. ☐

Corollary 1.12. *Consider a random sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined on the probability space $(\Omega, \mathcal{F}, P)$, with its natural filtration $\mathcal{F}_\bullet$. Let $a \in \mathbb{R}$ be a constant, and consider two random sequences $Y : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and $Z : \Omega \to \mathbb{R}^{\mathbb{N}}$ generated by X, such that for each $n \in \mathbb{N}$,*

$$Y_n \triangleq (X_n - a)_+ = (X_n \vee a) - a, \qquad\qquad Z_n \triangleq X_n \wedge a.$$

i_ If X is a submartingale with respect to $\mathcal{F}_\bullet$, then so is Y with respect to $\mathcal{F}_\bullet$.

ii_ If X is a supermartingale with respect to $\mathcal{F}_\bullet$, then so is Z with respect to $\mathcal{F}_\bullet$.

1.1 Stopping Times

Consider a discrete filtration $\mathcal{F}_\bullet = (\mathcal{F}_n \subseteq \mathcal{F} : n \in \mathbb{Z}_+)$.

Definition 1.13. A positive integer valued, possibly infinite, random variable N is said to be a **random time** with respect to the filtration $\mathcal{F}_\bullet$, if the event $\{N = n\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$. If $P\{N < \infty\} = 1$, then the random time N is said to be a **stopping time**.

Definition 1.14. A random sequence $H : \Omega \to \mathbb{R}^{\mathbb{N}}$ is **predictable** with respect to the the filtration $\mathcal{F}_\bullet$, if $\sigma(H_n) \subseteq \mathcal{F}_{n-1}$ for each $n \in \mathbb{N}$. Further, we define

$$(H \cdot X)_n \triangleq \sum_{m=1}^{n} H_m (X_m - X_{m-1}).$$

Theorem 1.15. *Consider a supermartingale sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ and a predictable sequence $H : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with respect to a filtration $\mathcal{F}_\bullet$, where each H_n is non-negative and bounded. Then the random sequence $Y : \Omega \to \mathbb{R}^N$ defined by $Y_n = (H \cdot X)_n$ for each $n \in \mathbb{N}$ is a supermartingale with respect to $\mathcal{F}_\bullet$.*

Proof. It follows from the definition,

$$\mathbb{E}[(H \cdot X)_{n+1} | \mathcal{F}_n] = \mathbb{E}[H_{n+1}(X_{n+1} - X_n) + (H \cdot X)_n | \mathcal{F}_n] = H_{n+1}(\mathbb{E}[X_{n+1}|\mathcal{F}_n] - X_n) + (H \cdot X)_n \leqslant (H \cdot X)_n.$$

□

1.2 Stopped process

Definition 1.16. Consider a discrete stochastic process $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ adapted to a discrete filtration $\mathcal{F}_\bullet$. Let $T : \Omega \to \mathbb{N}$ be a random time for the filtration $\mathcal{F}_\bullet$, then the **stopped process** $X^T : \Omega \to \mathbb{R}^{\mathbb{N}}$ is defined for each $n \in \mathbb{N}$ as

$$X_n^T \triangleq X_{T \wedge n} = X_n \mathbb{1}_{\{n \leqslant T\}} + X_T \mathbb{1}_{\{n > T\}}.$$

Proposition 1.17. *Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be a martingale with a discrete filtration $\mathcal{F}_\bullet$. If $T : \Omega \to \mathbb{N}$ is an integer random time for the filtration $\mathcal{F}_\bullet$, then the stopped process $(X_{T \wedge n} : n \in \mathbb{N})$ is a martingale.*

Proof. Consider a random sequence $H : \Omega \to \{0,1\}^{\mathbb{N}}$ defined by $H_n \triangleq \mathbb{1}_{\{n \leqslant T\}}$ for each $n \in \mathbb{N}$. Then H is a non-negative and bounded sequence. Further H is predictable with respect to $\mathcal{F}_\bullet$, since the event

$$\{n \leqslant T\} = \{T > n-1\} = \{T \leqslant n-1\}^c = (\cup_{i=0}^{n-1}\{T = i\})^c = \cap_{i=0}^{n-1}\{T \neq i\} \in \mathcal{F}_{n-1}.$$

In terms of the non-negative, predictable, and bounded sequence H, we can write the stopped process as

$$X_{T \wedge n} = X_0 + \sum_{m=1}^{T \wedge n}(X_m - X_{m-1}) = X_0 + \sum_{m=1}^{n} \mathbb{1}_{\{m \leqslant T\}}(X_m - X_{m-1}) = X_0 + (H \cdot X)_n.$$

From the previous theorem, it follows that $X_{T \wedge n}$ is a martingale, and we have $\mathbb{E}X_{T \wedge n} = \mathbb{E}X_{T \wedge 1} = \mathbb{E}X_1$. □

Remark 2. For any martingale $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ and a stopping time $T : \Omega \to \mathbb{N}$ adapted to $\mathcal{F}_\bullet$, we have $\mathbb{E}X_{T \wedge n} = \mathbb{E}X_1$, for all $n \in \mathbb{N}$. It is immediate that stopped process converges almost surely to X_T, i.e.

$$P\left\{\lim_{n \in \mathbb{N}} X_{T \wedge n} = X_T\right\} = 1.$$

This is true because $T < \infty$ almost surely. We are interested in knowing under what conditions will we have convergence in mean.

Theorem 1.18 (Martingale stopping theorem). *Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be a martingale and $T : \Omega \to \mathbb{N}$ be a stopping time adapted to a discrete filtration $\mathcal{F}_\bullet$. Then, the random variable X_T is integrable and the stopped process $X_{T\wedge n}$ converges in mean to X_T, i.e.*

$$\lim_{n\in\mathbb{N}} \mathbb{E}X_{T\wedge n} = \mathbb{E}X_T = \mathbb{E}X_1,$$

if either of the following conditions holds true.

(i) T is bounded,

(ii) $X_{T\wedge n}$ is uniformly bounded,

(iii) $\mathbb{E}T < \infty$, and for some real positive K, we have $\sup_{n\in\mathbb{N}} \mathbb{E}[|X_{n+1} - X_n||\mathcal{F}_n] < K$.

Proof. We show this is true for all three cases.

(i) Let K be the bound on T then for all $n \geqslant K$, we have $X_{T\wedge n} = X_T$, and hence it follows that

$$\mathbb{E}X_1 = \mathbb{E}X_{T\wedge n} = \mathbb{E}X_T, \qquad \text{for all } n \geqslant K.$$

(ii) Dominated convergence theorem implies the result.

(iii) Since T is integrable and $X_{T\wedge n} \leqslant |X_1| + KT$, we observe that $X_{T\wedge n}$ is bounded by an integrable random variable. The result follows from dominated convergence theorem.

□

Corollary 1.19 (Wald's Equation). *If T is a stopping time for the discrete* i.i.d. *random sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ with $\mathbb{E}|X| < \infty$ and $\mathbb{E}T < \infty$, then*

$$\mathbb{E}\sum_{i=1}^{T} X_i = \mathbb{E}T\mathbb{E}X.$$

Proof. Let $\mu = \mathbb{E}X$ and define a random sequence $Z : \Omega \to \mathbb{R}^{\mathbb{N}}$ such that $Z_n \triangleq \sum_{i=1}^{n}(X_i - \mu)$ for each $n \in \mathbb{N}$, Then Z is a martingale adapted to natural filtration of X, and hence from the Martingale stopping theorem, we have $\mathbb{E}Z_T = \mathbb{E}Z_1 = 0$. However, we observe that

$$\mathbb{E}[Z_T] = \mathbb{E}\sum_{i=1}^{T} X_i - \mu\mathbb{E}T.$$

Observe that condition (iii) for Martingale stopping theorem to hold can be directly verified. Hence the result follows. □