Lecture-04: Stochastic Processes

1 Stochastic Processes

Definition 1.1. For an arbitrary index set T and a real valued function $x \in \mathbb{R}^T$, the projection operator $\pi_t : \mathbb{R}^T \to \mathbb{R}$ is defined as $\pi_t(x) \triangleq x_t$ for any $x \in \mathbb{R}^T$.

Remark 1. Recall that $\pi_t^{-1}(-\infty, x] = \bigtimes_{s \in T}(-\infty, x_s]$ where $x_s = x$ for $s = t$ and $x_s = \infty$ for all $s \neq t$. That is, we can write $\pi_t^{-1}(-\infty, x_t] = (-\infty, x_t] \bigtimes_{s \in T: s \neq t} \mathbb{R}$ for any $x_t \in \mathbb{R}$.

Definition 1.2 (Random process). Let $(\Omega, \mathcal{F}, P)$ be a probability space. For an arbitrary index set T and state space $\mathcal{X} \subseteq \mathbb{R}$, a map $X : \Omega \to \mathcal{X}^T$ is called a **random process** if the projections $X_t : \Omega \to \mathcal{X}$ defined by $X_t(\omega) \triangleq (\pi_t \circ X)(\omega)$ are random variables on the given probability space. For each outcome $\omega \in \Omega$, we have a function $X(\omega) \in \mathcal{X}^T$ called the **sample path** or the **sample function** of the process X.

Remark 2. A random process X defined on probability space $(\Omega, \mathcal{F}, P)$ with index set T and state space $\mathcal{X} \subseteq \mathbb{R}$, can be thought of as
(a) a map $X : \Omega \times T \to \mathcal{X}$,
(b) a map $X : T \to \mathcal{X}^\Omega$, i.e. a collection of random variables $X_t : \Omega \to \mathcal{X}$ for each time $t \in T$,
(c) a map $X : \Omega \to \mathcal{X}^T$, i.e. a collection of sample functions $X(\omega) \in \mathcal{X}^T$ for each random outcome $\omega \in \Omega$.

1.1 Classification

State space $\mathcal{X}$ can be countable or uncountable, corresponding to discrete or continuous valued process. If the index set T is countable, the stochastic process is called **discrete**-time stochastic process or random sequence. When the index set T is uncountable, it is called **continuous**-time stochastic process. The index set T doesn't have to be time, if the index set is space, and then the stochastic process is spatial process. When $T = \mathbb{R}^n \times [0, \infty)$, stochastic process X is a spatio-temporal process.

1.2 Measurability

Definition 1.3. A random map $X : \Omega \to \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$ is called a $\mathcal{F}$-**measurable** random process, if the projections $X_t \triangleq \pi_t \circ X$ are $\mathcal{F}$-measurable random variables for all $t \in T$.

Remark 3. A random process $X : \Omega \to \mathcal{X}^T$ is $\mathcal{F}$-measurable, if the set of outcomes $A_{X_t}(x_t) \triangleq X_t^{-1}(-\infty, x_t] \in \mathcal{F}$ for all $t \in T$ and $x_t \in \mathbb{R}$.

Definition 1.4. The **event space generated by a random process** $X : \Omega \to \mathcal{X}^T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is given by

$$\sigma(X) \triangleq \sigma(A_{X_t}(x) : t \in T, x \in \mathbb{R}).$$

Definition 1.5. For a random process $X : \Omega \to \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$, we define the projection of X onto components $S \subseteq T$ as the random vector $X_S : \Omega \to \mathcal{X}^S$, where $X_S \triangleq (X_s : s \in S)$.

Remark 4. The $\mathcal{F}$-measurability of process X implies that for any countable set $S \subseteq T$, we have $A_{X_S}(x_S) \triangleq \cap_{s \in S} A_{X_s}(x_s) \in \mathcal{F}$ for $x_S \in \mathcal{X}^S$.

Definition 1.6. We can define $A_X(x) \triangleq \cap_{t \in T} A_{X_t}(x_t) = \{\omega \in \Omega : (\pi_t \circ X)(\omega) \leqslant \pi_t(x)\}$ for any $x \in \mathbb{R}^T$.

Remark 5. However, $A_X(x)$ is guaranteed to be an event only when $S \triangleq \{t \in T : \pi_t(x) < \infty\}$ is a countable set. In this case,

$$A_X(x) = \cap_{t \in T} A_{X_t}(x_t) = \cap_{s \in S} A_{X_s}(x_s) = A_{X_S}(x_S) \in \mathcal{F}.$$

Remark 6. For any finite subset $S \subseteq T$ and real vector $x \in \mathbb{R}^T$ such that $x_t = \infty$ for any $t \notin S$, we define a set

$$B(x) \triangleq \left\{ y \in \mathbb{R}^T : y_t \leqslant x_t \right\} = \bigtimes_{t \in T} (-\infty, x_t] = \bigtimes_{s \in S} (-\infty, x_s] \bigtimes_{t \notin S} \mathbb{R} = \cap_{t \in T} \pi_t^{-1}(-\infty, x_t].$$

The measurability of the random process X implies that for any such set $B(x)$, we have

$$A_X(x) = X^{-1}(B(x)) = \cap_{t \in T} (X^{-1} \circ \pi_t^{-1})(-\infty, x_t] = \cap_{t \in T} X_t^{-1}(-\infty, x_t] = \cap_{s \in S} X_s^{-1}(-\infty, x_s] \in \mathcal{F}.$$

Example 1.7 (Bernoulli sequence). Consider a sample space $\{H,T\}^{\mathbb{N}}$. We define a mapping $X : \Omega \to \{0,1\}^{\mathbb{N}}$ such that $X_n(\omega) = \mathbb{1}_{\{H\}}(\omega_n) = \mathbb{1}_{\{\omega_n = H\}}$. The map X is an $\mathcal{F}$-measurable random sequence, if each $X_n : \Omega \to \{0,1\}$ is a bi-variate $\mathcal{F}$-measurable random variable on the probability space $(\Omega, \mathcal{F}, P)$. Therefore, the event space $\mathcal{F}$ must contain the event space generated by events $E_n \triangleq \{\omega \in \Omega : X_n(\omega) = 1\} = \{\omega \in \Omega : \omega_n = H\} \in \mathcal{F}$. That is,

$$\sigma(X) = \sigma(E_n : n \in \mathbb{N}).$$

1.3 Distribution

Definition 1.8. For a random process $X : \Omega \to \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$, we define a **finite dimensional distribution** $F_{X_S} : \mathbb{R}^S \to [0,1]$ for a finite $S \subseteq T$ by

$$F_{X_S}(x_S) \triangleq P(A_{X_S}(x_S)) = P(\cap_{s \in S} A_{X_s}(x_s)), \quad x_S \in \mathbb{R}^S.$$

Example 1.9. Consider a probability space $(\Omega, \mathcal{F}, P)$ defined by the sample space $\Omega = \{H,T\}^{\mathbb{N}}$, the event space $\mathcal{F} \triangleq \sigma(E_n : n \in \mathbb{N})$ where $E_n = \{\omega \in \Omega : \omega_n = H\}$, and the probability measure $P : \mathcal{F} \to [0,1]$ defined by

$$P(\cap_{i \in F} E_i) = p^{|F|}, \text{ for all finite } F \subseteq \mathbb{N}.$$

Let $X : \Omega \to \{0,1\}^{\mathbb{N}}$ defined as $X_n(\omega) = \mathbb{1}_{E_n}(\omega)$ for all outcomes $\omega \in \Omega$ and $n \in \mathbb{N}$. For this random sequence, we can obtain the finite dimensional distribution $F_{X_S} : \mathbb{R}^S \to [0,1]$ for any finite $S \subseteq T$ and $x \in \mathbb{R}^S$ in terms of $U \triangleq \{i \in S : x_i < 0\}$ and $V \triangleq \{i \in S : x_i \in [0,1)\}$, as

$$F_{X_S}(x) = \begin{cases} 1, & U \cup V = \emptyset, \\ (1-p)^{|V|}, & U = \emptyset, V \neq \emptyset, \\ 0, & U \neq \emptyset. \end{cases} \tag{1}$$

To define a measure on a random process, we can either put a measure on sample paths $(X(\omega) \in \mathcal{X}^T : \omega \in \Omega)$, or equip the collection of random variables $(X_t \in \mathcal{X}^\Omega : t \in T)$ with a joint measure. Either way, we are interested in identifying the joint distribution $F : \mathbb{R}^T \to [0,1]$. To this end, for any $x \in \mathbb{R}^T$, we need to know

$$F_X(x) \triangleq P\left(\bigcap_{t \in T} \{\omega \in \Omega : X_t(\omega) \leqslant x_t\} \right) = P(\bigcap_{t \in T} X_t^{-1}(-\infty, x_t]) = P \circ X^{-1} \bigtimes_{t \in T} (-\infty, x_t].$$

First of all, we don't know whether $A_X(x)$ is an event when T is uncountable. Though, we can verify that $A_X(x) \in \mathcal{F}$ for $x \in \mathbb{R}^T$ such that $\{t \in T : x_t < \infty\}$ is countable. Second, even for a simple independent process with countably infinite T, any function of the above form would be zero if x_t is finite for all $t \in T$. That is, for any finite set $S \subseteq T$, we focus on the events $A_S(x_S)$ and their probabilities. However, these are precisely the finite dimensional distributions. Set of all finite dimensional distributions of a stochastic process $X : \Omega \to \mathcal{X}^T$ characterizes its distribution completely. Simpler characterizations of a stochastic process X are in terms of its moments. That is, the first moment such as mean, and the second moment such as correlations and covariance functions.

$$m_X(t) \triangleq \mathbb{E}X_t, \qquad R_X(t,s) \triangleq \mathbb{E}X_t X_s, \qquad C_X(t,s) \triangleq \mathbb{E}(X_t - m_X(t))(X_s - m_X(s)).$$

Example 1.10. Consider a probability space $(\Omega,\mathcal{F},P)$ defined by the sample space $\Omega = \{H,T\}^{\mathbb{N}}$ and the event space $\mathcal{F} \triangleq \sigma(E_n : n \in \mathbb{N})$ where $E_n = \{\omega \in \Omega : \omega_n = H\}$. Let $X : \Omega \to \{0,1\}^{\mathbb{N}}$ defined as $X_n(\omega) = \mathbb{1}_{E_n}(\omega)$ for all outcomes $\omega \in \Omega$ and $n \in \mathbb{N}$. For this random sequence, if we are given the finite dimensional distribution $F_{X_S} : \mathbb{R}^S \to [0,1]$ for any finite $S \subseteq T$ and $x \in \mathbb{R}^S$ in terms of $U \triangleq \{i \in S : x_i < 0\}$ and $V \triangleq \{i \in S : x_i \in [0,1)\}$, as defined in Eq. (1). Then, we can find the probability measure $P : \mathcal{F} \to [0,1]$ is given by

$$P(\cap_{i\in F} E_i) = p^{|F|}, \text{ for all finite } F \subseteq \mathbb{N}.$$

Let $q \triangleq (1-p)$, then the probability of observing m heads and r tails is given by $p^m q^r$. We can easily compute the mean, the auto-correlation, and the auto-covariance functions for this independent Bernoulli process

$$m_X(n) = \mathbb{E}X_n = p, \qquad R_X(m,n) = \mathbb{E}X_m X_n = \mathbb{E}X_m \mathbb{E}X_n = p^2, \qquad C_x(m,n) = 0.$$

1.4 Independence

Definition 1.11. A stochastic process $X : \Omega \to \mathcal{X}^T$ is said to be **independent** if for all finite subsets $S \subseteq T$, the finite collection of events $\{\{X_s \leqslant x_s\} : s \in S\}$ are independent. That is, we have

$$F_{X_S}(x_S) = P(\cap_{s\in S}\{X_s \leqslant x_s\}) = \prod_{s\in S} P\{X_s \leqslant x_s\} = \prod_{s\in S} F_{X_s}(x_s).$$

Remark 7. Independence of a random process is equivalent to factorization of any finite dimensional distribution function into product of individual marginal distribution functions.

Example 1.12. Consider a probability space $(\Omega,\mathcal{F},P)$ defined by the sample space $\Omega = \{H,T\}^{\mathbb{N}}$, the event space $\mathcal{F} \triangleq \sigma(E_n : n \in \mathbb{N})$ where $E_n = \{\omega \in \Omega : \omega_n = H\}$, and the probability measure $P : \mathcal{F} \to [0,1]$ defined by

$$P(\cap_{i\in F} E_i) = p^{|F|}, \text{ for all finite } F \subseteq \mathbb{N}.$$

Then, we observe that the random sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ defined by $X_n(\omega) \triangleq \mathbb{1}_{E_n}(\omega)$ for all outcomes $\omega \in \Omega$ and $n \in \mathbb{N}$, is independent.

Definition 1.13. Two stochastic processes $X : \Omega \to \mathcal{X}^{T_1}, Y : \Omega \to \mathcal{Y}^{T_2}$ are **independent**, if the corresponding event spaces $\sigma(X),\sigma(Y)$ are independent. That is, for any $x \in \mathbb{R}^{S_1}, y \in \mathbb{R}^{S_2}$ for finite $S_1 \subseteq T_1, S_2 \subseteq T_2$, the events $A_{S_1}(x) \triangleq \cap_{s\in S_1} X_s^{-1}(-\infty,x_s]$ and $B_{S_2}(y) \triangleq \cap_{s\in S_2} Y_s^{-1}(-\infty,y_s]$ are independent. That is, the joint finite dimensional distribution of X and Y factorizes, and

$$P(A_{S_1}(x) \cap B_{S_2}(y)) = P(A_{S_1}(x))P(B_{S_2}(y)) = F_{X_{S_1}}(x)F_{Y_{S_2}}(y), \quad x \in \mathbb{R}^{S_1}, y \in \mathbb{R}^{S_2}.$$

1.5 Filtration

Let $(\Omega,\mathcal{F},P)$ be a probability space.

Definition 1.14. A net of event spaces denoted $\mathcal{F}_\bullet = (\mathcal{F}_t \subseteq \mathcal{F} : t \in T)$ is called a **filtration** if the index set T is totally ordered and the net is nondecreasing, that is $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leqslant t$.

Definition 1.15. Consider a real-valued random process X indexed by the ordered set T on the probability space $(\Omega,\mathcal{F},P)$. The process X is called **adapted** to the filtration $\mathcal{F}_\bullet$, if for each $t \in T$, we have $\sigma(X_t) \subseteq \mathcal{F}_t$ or $X_t^{-1}(-\infty,x] \in \mathcal{F}_t$ for each $x \in \mathbb{R}$.

We will consider any random process $X : \Omega \to \mathcal{X}^T$ defined on this probability space with state space $\mathcal{X} \subseteq \mathbb{R}$ and ordered index set $T \subseteq \mathbb{R}$ considered as time.

Definition 1.16. For the random process $X : \Omega \to \mathcal{X}^T$, we define the event space generated by all random variables until time t as $\mathcal{G}_t \triangleq \sigma(X_s, s \leqslant t)$.

Remark 8. The collection of event spaces $\mathcal{G}_\bullet = (\mathcal{G}_t : t \in T)$ is a filtration.

Definition 1.17. The natural filtration associated with a random process $X : \Omega \to \mathcal{X}^T$ is given by $\mathcal{G}_\bullet = (\mathcal{G}_t : t \in T)$ where $\mathcal{G}_t \triangleq \sigma(X_s, s \leqslant t)$.

Remark 9. Any random process X is adapted to its natural filtration.

Remark 10. For a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, the natural filtration is a sequence $\mathcal{G}_\bullet = (\mathcal{G}_n \subseteq \mathcal{F} : n \in \mathbb{N})$ of event spaces $\mathcal{G}_n \triangleq \sigma(X_1, \dots, X_n)$ for all $n \in \mathbb{N}$.

Example 1.18. For a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with step size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined by $S_n \triangleq \sum_{i=1}^n X_i$ for all $n \in \mathbb{N}$, the natural filtration of the random walk is identical to that of the step size sequence. That is, $\sigma(S_1, \dots, S_n) = \sigma(X_1, \dots, X_n)$ for all $n \in \mathbb{N}$. This follows from the fact that for all $n \in \mathbb{N}$, we can can write $S_j = \sum_{i=1}^j X_i$ and $X_j = S_j - S_{j-1}$ for all $j \in [n]$. That is, there is a bijection between $(X_1, \dots, X_n)$ and $(S_1, \dots, S_n)$.

Remark 11. If the random sequence X is independent, then the random sequence $(X_{n+j} : j \in \mathbb{N})$ is independent of the event space $\sigma(X_1, \dots, X_n)$.

Remark 12. Let $X : \Omega \to \mathcal{X}^T$ be an independent process with the associated natural filtration $\mathcal{G}_\bullet = (\mathcal{G}_t : t \in T)$ for an ordered index set T. Then for any $t > s$ and events $A \in \mathcal{G}_s$, the random variable X_t is independent of the event A. This is just a fancy way of saying X_t is independent of $\sigma(X_u, u \leqslant s)$. Hence, for any random variable $Y \in \mathcal{G}_s$, we have

$$\mathbb{E}[X_t Y | \mathcal{G}_s] = Y\mathbb{E}[X_t].$$

1.6 Progressive measurability

For continuous-time processes, where the time t ranges over an arbitrary index set $T \subseteq \mathbb{R}$, the property of being adapted is too weak to be helpful in many situations. Instead, we need to consider measurability of the process as a map $X : T \times \Omega \to \mathbb{R}$. To this end, we first define measurability on the product spaces.

Definition 1.19. Let $(S, \mathcal{S})$ and $(V, \mathcal{V})$ be two measurable spaces. The product measurable space denoted $(S \times V, \mathcal{S} \otimes \mathcal{V})$ is defined as

$$\mathcal{S} \otimes \mathcal{V} \triangleq \sigma(A \times B : A \in \mathcal{S}, B \in \mathcal{V}).$$

Definition 1.20. For a product measurable space $(S \times V, \mathcal{S} \otimes \mathcal{V})$, we define projections $\pi_S(A \times B) = A$ and $\pi_V(A \times B) = B$ for any $A \times B \in \mathcal{S} \otimes \mathcal{V}$.

Definition 1.21. For a random process $X : T \times \Omega \to \mathcal{X}$ and any time $s \in T$, we can define a stopped process $X^s : T \times \Omega \to \mathcal{X}$ such that $X^s_t \triangleq X_{t \wedge s}$ for all $t \in T$.

Definition 1.22. A process $X : T \times \Omega \to \mathbb{R}$ adapted to filtration $\mathcal{F}_\bullet$ is **progressive** or **progressively measurable**, if stopped process X^s is $\mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$ measurable for all $s \in T$.

Remark 13. Since $\pi_\Omega \circ (X^{-1}(-\infty, x] \cap (\{t\} \times \Omega)) = X_t^{-1}(-\infty, x] \in \mathcal{F}_t$, every progressively measurable process is adapted and jointly measurable.

Lemma 1.23. *When T is countable, every adapted process is progressive.*

Proof. It suffices to show this for countable $T = \mathbb{N}$. Let $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ be a real valued process adapted to filtration $\mathcal{F}_\bullet$, and X^m be a stopped process for $m \in \mathbb{N}$. We observe that the inverse map $(X^m)^{-1}(-\infty, x] \triangleq \{(n, \omega) : n \leqslant m, X(n, \omega) \leqslant x\} = \cup_{n \in [m]}(\{n\} \times X_n^{-1}(-\infty, x]) \in \mathcal{B}((-\infty, m]) \otimes \mathcal{F}_m$. □

Definition 1.24. A set $S \subseteq T \times \Omega$ is said to be **progressive** if its indicator function $\mathbb{1}_S$ is progressive. Equivalently, $S \cap (-\infty, s] \times \Omega \in \mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$ for all $s \in T$.

Proposition 1.25. *The progressively measurable sets form a σ-algebra.*

Proof. By definition product event space $\mathcal{B}((-\infty,s]) \otimes \mathcal{F}_s$ is a σ-algebra for all $s \in T$. We define the collection

$$\mathcal{G} \triangleq \{S \subseteq T \times \Omega : S \cap (-\infty,s] \times \Omega \in \mathcal{B}((-\infty,s]) \otimes \mathcal{F}_s \text{ for all } s \in T\}.$$

We need to show the following three conditions for $\mathcal{G}$ to be a σ-algebra.

(i) It is easy to see that $T \times \Omega \in \mathcal{G}$ since $(T \times \Omega) \cap ((-\infty,s] \times \Omega) = (-\infty,s] \times \Omega \in \mathcal{B}((-\infty,s]) \otimes \mathcal{F}_s$ for all $s \in T$ by definition.

(ii) Let $S \in \mathcal{G}$, then we will show that $S^c \in \mathcal{G}$. Let $s \in T$, then using the fact that $\mathcal{B}((-\infty,s]) \otimes \mathcal{F}_s$ is a σ-algebra, it follows that $S^c \cap ((-\infty,s] \times \Omega) = (S \cap (-\infty,s] \times \Omega)^c \cap ((-\infty,s] \times \Omega) \in \mathcal{B}((-\infty,s]) \otimes \mathcal{F}_s$.

(iii) Let $S \in \mathcal{G}^{\mathbb{N}}$, $s \in T$, and $S_n \cap (-\infty,s] \times \Omega \in \mathcal{B}((-\infty,s]) \otimes \mathcal{F}_s$ for all $n \in \mathbb{N}$, then it follows from the distributive property of intersections and the closure of $\mathcal{B}((-\infty,s]) \otimes \mathcal{F}_s$ under countable unions, that $\cup_{n\in\mathbb{N}} S_n \in \mathcal{G}$.

□

Proposition 1.26. *A stochastic process is progressive iff it is measurable with respect to progressive σ-algebra.*

Proof. Let $X : \Omega \to \mathcal{X}^T$ be a random process adapted to a filtration $\mathcal{F}_\bullet$. Let X be progressive and fix $s \in T$ and $x \in \mathbb{R}$, then we show that any event generated by the stopped process X^s is progressive. Indeed, we observe that $(X^s)^{-1}(-\infty,x] \cap ((-\infty,u] \times \Omega) = (X^{s\wedge u})^{-1}(-\infty,x] \in \mathcal{B}((-\infty,u]) \otimes \mathcal{F}_u$ for all $u \in T$.

Conversely, if we assume that any event generated by X is progressive, then $X^{-1}(-\infty,x] \cap ((-\infty,s] \cap \Omega) = (X^s)^{-1}(-\infty,x] \in \mathcal{B}((-\infty,s]) \otimes \mathcal{F}_s$ for all $s \in T$ and $x \in \mathbb{R}$. It follows that X is progressive. □

Proposition 1.27. *Every adapted process with right-continuous sample paths is progressively measurable.*

Theorem 1.28. *Every measurable and adapted process has a progressively measurable modification.*