Lecture-07: Renewal Process

1 Counting processes

Definition 1.1. A stochastic process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a **counting process** if
i_ $N_0 = 0$, and
ii_ for each outcome $\omega \in \Omega$, the map $t \mapsto N_t$ is non-decreasing, integer valued, and right continuous.

Lemma 1.2. *A counting process has finitely many jumps in a finite interval* $(0, t]$.

Definition 1.3. A simple counting process has discontinuities of unit size.

Definition 1.4. The nth point of discontinuity of a simple counting process N_t is called the n**th arrival instant** and denoted by $S_n : \Omega \to \mathbb{R}_+$ such that

$$S_0 \triangleq 0, \qquad S_n \triangleq \inf\{t \geqslant 0 : N_t \geqslant n\}, \; n \in \mathbb{N}.$$

The random sequence of arrival instants is denoted by $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. The **inter arrival time** between $(n-1)$th and nth arrival is denoted by $X_n \triangleq S_n - S_{n-1}$.

Exercise 1.5. Show that $P\{X_n \leqslant 0\} = 0$ for simple counting processes.

Remark 1. The arrival sequence S is non-decreasing for each outcome $\omega \in \Omega$, since inf is non-decreasing for decreasing sets. That is, $S_n \leqslant S_{n+1}$ for all $n \in \mathbb{N}$.

Lemma 1.6 (Inverse processes). *Inverse of a simple counting process N is its corresponding arrival process S. That is,*

$$\{S_n \leqslant t\} = \{N_t \geqslant n\}. \tag{1}$$

Proof. Let $\omega \in \{S_n \leqslant t\}$. Since N is a non-decreasing process, we have $N_t \geqslant N_{S_n} = n$. Conversely, let $\omega \in \{N_t \geqslant n\}$, then it follows from definition that $S_n \leqslant t$. ☐

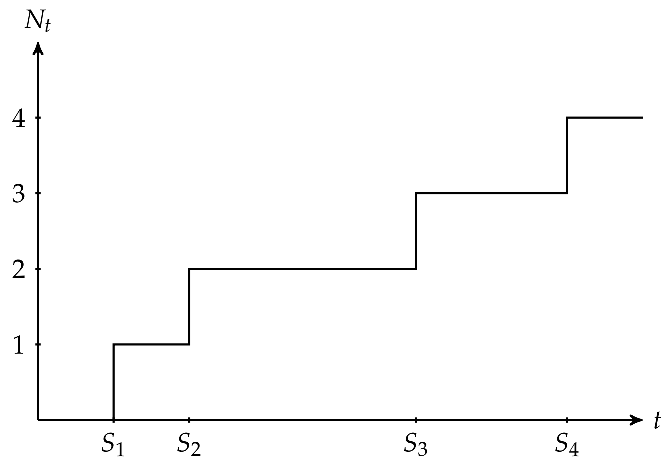

Figure 1: Sample path of a simple counting process.

Remark 2. Let $\mathcal{F}_\bullet = (\mathcal{F}_s : s \geqslant 0)$ be the natural filtration associated with the counting process N, that is $\mathcal{F}_t \triangleq \sigma(N_s, s \in [0, t])$. Then $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a sequence of random times adapted to filtration $\mathcal{F}_\bullet$.

Corollary 1.7. *The probability mass function for the counting process N_t at time t can be written in terms of distribution functions of arrival sequence S as*

$$P\{N_t = n\} = F_{S_n}(t) - F_{S_{n+1}}(t).$$

Proof. The event $\{N_t \geqslant n\}$ is the union of two disjoint events $\{N_t = n\} \cup \{N_t \geqslant n+1\}$, and the result follows from the probability of disjoint unions. □

Definition 1.8. A **point process** is a collection $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ of randomly distributed points, such that $\lim_{n\to\infty} |S_n| = \infty$. A point process is simple if the points are distinct. Let $N(\emptyset) = 0$ and denote the number of points in a measurable set $A \in \mathcal{B}(\mathcal{X})$ by

$$N(A) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \in A\}}.$$

Then $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ is called a **counting process** for the simple point process S.

Remark 3. When $\mathcal{X} = \mathbb{R}_+$, one can order these points of S as an increasing sequence such that $S_1 < S_2 < \dots$, and denote the number of points in a half-open interval $(0,t]$ by

$$N_t \triangleq N(0,t] = \sum_{n\in\mathbb{N}} \mathbb{1}_{(0,t]}(S_n) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}.$$

Remark 4. General point processes in higher dimension don't have any inter-arrival time interpretation.

Exercise 1.9. Show that $P\{X_n \leqslant 0\} = 0$ for simple point processes on $\mathbb{R}_+$.

2 Renewal processes

Definition 2.1 (Renewal Instants). Consider an *i.i.d.* sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ of **inter-renewal times** and denote the nth **renewal instant** by $S_n \triangleq \sum_{i=1}^{n} X_i$ for all $n \in \mathbb{N}$, and $S_0 = 0$. The random sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is called sequence of renewal instants or renewal sequence.

Remark 5. We interpret X_n as the time between the $(n-1)$th and the nth renewal event, with a common distribution F. If $F(0) = 1$, then it is a trivial process. Hence we will often assume that $F(0) < 1$ to imply a non-degenerate renewal process.

Example 2.2 (Random walk). Random walk S on $\mathbb{R}_+$ with *i.i.d.* non-negative step-sizes $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a renewal sequence.

Example 2.3 (Markov chain). Let $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ be a discrete time homogeneous Markov chain X with state space $\mathcal{X}$. For $X_0 = x \in \mathcal{X}$ and defining $\tau_x^+(0) \triangleq 0$, let the recurrent times be defined inductively as

$$\tau_x^+(n) = \inf\left\{k > \tau_x^+(n-1) : X_k = x\right\}. \tag{2}$$

It follows from the strong Markov property of the process X, that $\tau_x^+ : \Omega \to \mathbb{R}_+^{\mathbb{Z}_+}$ is a renewal sequence.

Definition 2.4 (Renewal process). The associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ that counts number of renewal until time t with *i.i.d.* general inter-renewal times is called a **renewal process**, written as

$$N_t \triangleq \sup\{n \in \mathbb{Z}_+ : S_n \leqslant t\} = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}.$$

Definition 2.5. A renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is said to be **recurrent** if the inter-renewal time X_n is finite almost surely for every $n \in \mathbb{N}$, the process is called **transient** otherwise. A renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is said to be **positive recurrent** if the inter-renewal time X_n has finite mean.

Remark 6. We will mostly be interested in a positive recurrent renewal process, and hence we will often assume that the mean $\mu = \mathbb{E}X_1 = \int_{\mathbb{R}} xdF(x)$ is finite.

Definition 2.6. The process is said to be **periodic** with period d if the inter-renewal times $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ take values in a discrete set $\mathcal{X} \subseteq \{nd : n \in \mathbb{Z}_+\}$ and $d = \gcd(\mathcal{X})$ is the largest such number. Otherwise, if there is no such $d > 0$, then the process is said to be **aperiodic**. If the inter-arrival times is a periodic random variable, then the associated distribution function F is called **lattice**.

Lemma 2.7 (Finiteness). *For a renewal process with mean inter-renewal time $\mathbb{E}X_n > 0$, the number of renewals N_t in the time duration $(0,t]$ is a.s. finite for all $t > 0$.*

Proof. We are interested in knowing how many renewals occur per unit time. Since X_n is non-negative, we have $\mu = \mathbb{E}X_n = \mathbb{E}|X_n|$. Without any loss of generality, we assume that $\mu < \infty$. Then, from the L^1 strong law of large numbers, we know that

$$P\left\{\lim_{n\in\mathbb{N}} \frac{S_n}{n} = \mu\right\} = 1.$$

Since $\mu > 0$, we must have S_n growing arbitrarily large as n increases. Thus, S_n can be finite for at most finitely many n. Indeed for any finite t, we have the the following set inclusion

$$\bigcap_{n\in\mathbb{N}} \{N_t \geqslant n\} = \bigcap_{n\in\mathbb{N}} \{S_n \leqslant t\} \subseteq \bigcap_{n\in\mathbb{N}} \left\{\frac{S_n}{n} \leqslant \frac{t}{n}\right\} \subseteq \left\{\limsup_{n\in\mathbb{N}} \frac{S_n}{n} = 0\right\}.$$

Since $\mu > 0$, we obtain $\left\{\limsup_{n\in\mathbb{N}} \frac{S_n}{n} = 0\right\} \subseteq \left\{\lim_{n\in\mathbb{N}} \frac{S_n}{n} = \mu\right\}^c$, it follows that $P\{N_t = \infty\} = 0$ for any finite t. The result follows and $N_t = \max\{n \in \mathbb{N} : S_n \leqslant t\}$. ☐

Exercise 2.8. Show that for sequences $x \in \mathbb{R}^{\mathbb{N}}$ and $\alpha \in \mathbb{R}^{\mathbb{N}}$, if $x_n \leqslant \alpha_n$ for all $n \in \mathbb{N}$, then $\limsup_n x_n \leqslant \limsup_n \alpha_n$.

2.1 Delayed renewal processes

Many times in practice, we have a *delayed start* to a renewal process. That is, the renewal process has independent inter-renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, where the common distribution for X_n is F when $n \geqslant 2$, and the distribution of first inter-arrival time X_1 is G. Similar to the renewal process, the initial renewal instant is assumed to be $S_0 = 0$ and the nth renewal instant is $S_n \triangleq \sum_{i=1}^n X_i$ for all $n \in \mathbb{N}$. The associated counting process is called a **delayed renewal process** and denoted by $N_D : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. The following inverse relationship continues to hold between the counting process and the renewal sequence,

$$N_D(t) = \sup\{n \in \mathbb{N} : S_n \leqslant t\}. \tag{3}$$

Example 2.9 (Markov chain). Let $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ be a discrete time homogeneous Markov chain X with state space V. For $X_0 = x \in \mathcal{X}$ and $y \neq x$, let $\tau_y^+(0) \triangleq 0$, inductively define the recurrent times as

$$\tau_y^+(k) \triangleq \inf\left\{n > \tau_y^+(k-1) : X_n = y\right\}.$$

It follows from the strong Markov property of the process X, that $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{Z}_+}$ is a delayed renewal sequence.

Exercise 2.10. Consider a delayed renewal process with positive mean inter-renewal duration $\mathbb{E}X_n > 0$ for $n \geqslant 2$ and finite mean for the first renewal instant $\mathbb{E}X_1$. Show that the number of renewals $N_D(t)$ in the time duration $(0,t]$ is a.s. finite for all $t > 0$.