Lecture-08: Distribution and renewal functions

1 Convolution of distribution functions

Definition 1.1. For two distribution functions $F, G : \mathbb{R} \to [0,1]$ the convolution of F and G is a distribution function $F * G : \mathbb{R} \to [0,1]$ defined as

$$(F * G)(x) \triangleq \int_{y \in \mathbb{R}} F(x-y)dG(y),\ x \in \mathbb{R}.$$

Remark 1. We can verify that $F * G$ is indeed a distribution function. That is, the function $(F * G)$ is

(a) right continuous, i.e. $\lim_{x_n \downarrow x} (F * G)(x_n)$ exists,

(b) non-decreasing, i.e. $(F * G)(z) \geqslant (F * G)(x)$ for all $z \geqslant x$,

(c) having left limit of zero and right limit of unity, i.e. $\lim_{x \to -\infty} (F * G)(x) = 0, \lim_{x \to \infty} (F * G)(x) = 1$.

Part (a) and (c) can be verified by exchanging limit and integration using Monotone convergence theorem. Part (b) can be verified from monotonicity of integration.

Remark 2. We can verify that convolution is a symmetric and bi-linear operator. To show bi-linearity of convolution, we need to show for any two finite sets of distribution functions $(F_i : i \in [n])$ and $(G_j : j \in [m])$ and vectors $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m$, we have

$$*(\sum_{i \in [n]} \alpha_i F_i, \sum_{j \in [m]} \beta_j G_j) = (\sum_{i \in [n]} \alpha_i F_i) * (\sum_{j \in [m]} \beta_j G_j) = \sum_{i \in [n]} \sum_{j \in [m]} \alpha_i \beta_j (F_i * G_j).$$

This follows from the linearity of integration in its arguments. To show symmetry of convolution, we need to show for any two distribution functions (F, G), we have

$$F * G = *(F, G) = *(G, F) = G * F.$$

The symmetry can be verified by integration by parts and change of variables, since

$$\int_{y \in \mathbb{R}} F(x-y)dG(y) - \int_{x-y \in \mathbb{R}} dF(x-y)G(y) = F(x-y)G(y) \mid_{y=-\infty}^{y=\infty} = 0.$$

Lemma 1.2. *Let X and Y be two independent random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with distribution functions F and G respectively, then the distribution of $X + Y$ is given by $F * G$.*

Proof. The distribution function of sum $X + Y$ is given by $H : \mathbb{R} \to [0,1]$ where for any $z \in \mathbb{R}$,

$$H(z) = \mathbb{E}\mathbb{1}_{\{X+Y \leqslant z\}} = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{X+Y \leqslant z\}} | \sigma(Y)]] = \mathbb{E}[F(z - Y)] = \int_{y \in \mathbb{R}_+} F(z-y)dG(y).$$

□

Definition 1.3. Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be an *i.i.d.* random sequence defined on the probability space $(\Omega, \mathcal{F}, P)$ with the common distribution function F, then the distribution of $S_n \triangleq \sum_{i=1}^n X_i$ is denoted by F_n.

Remark 3. The distribution F_n is computed inductively as $F_n = F_{n-1} * F$ for all $n \geqslant 2$, where $F_1 = F$.

Remark 4. For a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with *i.i.d.* inter-renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having a common distribution $F : \mathbb{R}_+ \to [0,1]$, the distribution function of the nth renewal instant is the n-fold convolution F_n of the distribution function F.

Exercise 1.4 (Poisson process). For a renewal sequence S with the common distribution for *i.i.d.* inter-renewal times being $F(x) = 1 - e^{-\lambda x}$ for $x \in \mathbb{R}_+$, show by induction that the distribution of nth renewal instant at any time $t \in \mathbb{R}_+$ is

$$F_n(t) \triangleq \int_0^t \frac{\lambda^n s^{n-1}}{(n-1)!} e^{-\lambda s} ds.$$

One can observe that $F_1 = F$, and hence the base case of $n = 1$ holds. If the hypothesis is assumed to hold true for step $n-1$, then show that it holds for step n, where $F_n = F_{n-1} * F$.

Corollary 1.5. *The distribution function of nth arrival instant S_n for delayed renewal process is $G * F_{n-1}$.*

Corollary 1.6. *The distribution function of counting process $N^D : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ for the delayed renewal process is*

$$P\left\{N_t^D = n\right\} = P\left\{S_n \leqslant t\right\} - P\left\{S_{n+1} \leqslant t\right\} = (G * F_{n-1})(t) - (G * F_n)(t).$$

2 Renewal functions

Definition 2.1. Mean of the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is called the **renewal function** denoted by $m : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $m_t = \mathbb{E}[N_t]$ for all $t \in \mathbb{R}_+$.

Proposition 2.2. *Renewal function m for a renewal process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ can be written as $m_t = \sum_{n\in\mathbb{N}} F_n(t)$, where the distribution of the nth renewal instant is denoted by F_n for each $n \in \mathbb{N}$.*

Proof. Using the inverse relationship between counting process and the arrival instants, we can write

$$m_t = \mathbb{E}[N_t] = \sum_{n\in\mathbb{N}} P\left\{N_t \geqslant n\right\} = \sum_{n\in\mathbb{N}} P\left\{S_n \leqslant t\right\} = \sum_{n\in\mathbb{N}} F_n(t).$$

□

Example 2.3 (Poisson process). For a renewal sequence S with the common distribution for *i.i.d.* inter-renewal times being $F(x) = 1 - e^{-\lambda x}$ for $x \in \mathbb{R}_+$, the renewal function is

$$m_t = \sum_{n\in\mathbb{N}} F_n(t) = \int_0^t \lambda\Big(e^{-\lambda s} \sum_{n\in\mathbb{Z}_+} \frac{(\lambda s)^n}{s!}\Big) ds = \int_0^t \lambda ds = \lambda t.$$

Corollary 2.4. *The renewal function m_D for a delayed renewal process $N_D : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with distribution G for first inter-renewal times and F for other inter-renewal times, is given by $m_D = G + G * m$.*

Proof. We can write the renewal function for the delayed renewal process as

$$m_t^D = \mathbb{E}N_t^D = \sum_{n\in\mathbb{N}} (G * F_{n-1})(t) = G(t) + (G * m)(t).$$

□

Remark 5. If $G = F$, then $m = F + F * m$.

3 Laplace transform of distribution functions and renewal functions

Definition 3.1. The Laplace transform $\mathcal{L} : [0,1]^{\mathbb{R}} \to \mathbb{C}^{\mathbb{C}}$ for a distribution function $F : \mathbb{R} \to [0,1]$ is a map $\tilde{F} : \mathbb{C} \to \mathbb{C}$ defined for all $s \in \mathbb{C}$ such that $\left|\tilde{F}(s)\right| < \infty$, as

$$\mathcal{L}(F)(s) = \tilde{F}(s) \triangleq \int_{y\in\mathbb{R}} e^{-sy} dF(y).$$

Remark 6. If $X : \Omega \to \mathbb{R}$ is a random variable with distribution function F, then $\tilde{F}(s) = \mathbb{E}e^{-sX}$.

Lemma 3.2. *The Laplace transform of convolution of two distribution functions is product of Laplace transform of individual distribution functions.*

Proof. Let $F, G : \mathbb{R} \to [0,1]$ be two distribution functions such that $\mathcal{L}(F) = \tilde{F}$ and $\mathcal{L}(G) = \tilde{G}$, then

$$\mathcal{L}(F * G)(s) = \int_{x\in\mathbb{R}} e^{-sx} \int_{y\in\mathbb{R}} dF(x-y)dG(y) = \int_{y\in\mathbb{R}} e^{-sy} dG(y) \int_{x-y\in\mathbb{R}} e^{-s(x-y)} dF(x-y) = \mathcal{L}(F)\mathcal{L}(G).$$

□

Remark 7. Let the Laplace transform of the distribution functions of independent random variables X and Y be $\tilde{F}$ and $\tilde{G}$ respectively, then the Laplace transform of the distribution of $X + Y$ is $\tilde{F}\tilde{G}$.

Remark 8. Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be an *i.i.d.* random sequence with the Laplace transform $\tilde{F} = \mathcal{L}(F)$ of the common distribution function F, then the Laplace transform of the distribution of nth renewal instant $S_n \triangleq \sum_{i=1}^{n} X_i$ is given by $\mathcal{L}(F_n) = (\tilde{F})^n$.

Corollary 3.3. *Denoting the Laplace transform for the inter-renewal time distribution F by $\mathcal{L}(F) = \tilde{F}$, the Laplace transform of the renewal function m is given by*

$$\tilde{m}(s) = \frac{\tilde{F}(s)}{1 - \tilde{F}(s)}, \qquad \Re\{\tilde{F}(s)\} < 1.$$

Corollary 3.4. *For a delayed renewal process, we denote the Laplace transforms for the distributions of the first renewal time and the subsequent inter-renewal times by $\tilde{G} = \mathcal{L}(G)$ and $\tilde{F} = \mathcal{L}(F)$ respectively. The Laplace transform of the renewal function m_D for the delayed renewal process is*

$$\tilde{m}_D(s) = \frac{\tilde{G}(s)}{1 - \tilde{F}(s)}, \qquad \Re\{\tilde{F}(s)\} < 1.$$

Example 3.5 (Poisson process). The Laplace transform of an exponential distribution $F(x) = 1 - e^{-\lambda x}$ for $x \in \mathbb{R}_+$ is given by $\tilde{F}(s) = \frac{\lambda}{\lambda+s}$ for $\Re(s) > -\lambda$. For a renewal sequence S with the common distribution for *i.i.d.* inter-renewal times being the exponential distribution F, the Laplace transform for the renewal function is

$$\tilde{m}(s) = \frac{\tilde{F}(s)}{1 - \tilde{F}(s)} = \frac{\lambda}{s}, \qquad \Re(s) > -\lambda.$$

The Laplace transform for the distribution F_n is given by

$$\tilde{F}_n(s) = \left(1 + \frac{s}{\lambda}\right)^{-n}, \qquad \Re(s) > -\lambda.$$

Exercise 3.6. Invert the Laplace transform $\tilde{F}_n(s) = \left(1 + \frac{s}{\lambda}\right)^{-n}$ in the region of convergence $\Re(s) > -\lambda$ to obtain the distribution function F_n for the nth arrival instant of a Poisson process with rate λ.

Proposition 3.7. *For renewal process with $\mathbb{E}X_n > 0$, the renewal function is bounded for all finite times.*

Proof. Since we assumed that $P\{X_n = 0\} < 1$, it follow from continuity of probabilities that there exists $\alpha > 0$, such that $P\{X_n \geqslant \alpha\} = \beta > 0$. We can define bivariate random variables

$$\bar{X}_n = \alpha 1_{\{X_n \geqslant \alpha\}} \leqslant X_n.$$

Note that since X_i's are *i.i.d.* , so are $\bar{X}_i$'s. Each $\bar{X}_i$ takes values in $\{0, \alpha\}$ with probabilities $1 - \beta$ and β respectively. Let $\bar{N}_t$ denote the renewal process with inter-arrival times $\bar{X}_n$, with arrivals at integer multiples of α. Then for all sample paths, we have

$$N_t = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\sum_{i=1}^{n} X_i \leqslant t\}} \leqslant \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\sum_{i=1}^{n} \bar{X}_i \leqslant t\}} = \bar{N}_t.$$

Hence, it follows that $\mathbb{E}N_t \leqslant \mathbb{E}\bar{N}_t$, and we will show that $\mathbb{E}\bar{N}_t$ is finite. We can write the joint event of number of arrivals n_i at each arrival instant in $i\alpha$ for $i \in \{0, \dots, k-1\}$, as

$$\bigcap_{i=0}^{k-1} \{\bar{N}_{i\alpha} = n_i\} = \{X_1 < \alpha\} \bigcap_{i=0}^{k-1} \{X_{n_i+1} \geqslant \alpha\} \bigcap_{i=0}^{k-1} \bigcap_{j=2}^{n_i} \{X_{n_{i-1}+j} < \alpha\}.$$

It follows that the joint distribution of number of arrivals at first k arrival instants is

$$P\left(\bigcap_{i=0}^{k-1} \{\bar{N}_{i\alpha} = n_i\}\right) = (1-\beta) \prod_{i=0}^{k-1} (\beta)(1-\beta)^{n_i - 1}.$$

It follows that the number of arrivals is independent at each arrival instant $k\alpha$ and geometrically distributed with mean $1/\beta$ and $(1-\beta)/\beta$ for $k \in \mathbb{N}$ and $k = 0$ respectively. Thus, for all $t \geqslant 0$,

$$\mathbb{E}N_t \leqslant \mathbb{E}\bar{N}_t \leqslant \frac{\lceil \frac{t}{\alpha} \rceil}{\beta} \leqslant \frac{\frac{t}{\alpha} + 1}{\beta} < \infty.$$

□

Corollary 3.8. *For delayed renewal process with $\mathbb{E}X_n > 0$, the renewal function is bounded at all finite times.*