# Lecture-10: Regenerative Processes

## 1 Regenerative processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be a renewal sequence, with the associated inter-renewal sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. That is, the $n$th renewal instant is $S_n \triangleq \sum_{i=1}^n X_i$ for each $n \in \mathbb{N}$ and the number of renewals is $N_t \triangleq \sum_{n \in \mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$ until each time $t \in \mathbb{R}_+$.

**Definition 1.1.** Consider a stochastic process $Z : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ defined over the same probability space. The $n$th segment of the joint process $(N, Z) : \Omega \to (\mathbb{Z}_+ \times \mathbb{R})^{\mathbb{R}_+}$ is defined as the sample path in the $n$th inter-renewal duration, written $\zeta_n \triangleq (X_n, (Z_{S_{n-1}+t} : t \in [0, X_n)), \qquad n \in \mathbb{N}$.

**Definition 1.2.** The process $Z$ is regenerative over the renewal sequence $S$, if its segments $(\zeta_n : n \in \mathbb{N})$ are *i.i.d.* . The process $Z$ is delayed regenerative, if $S$ is a delayed renewal sequence and the segments $(\zeta_n : n \in \mathbb{N})$ of the joint process are independent with $(\zeta_n : n \geqslant 2)$ being identically distributed.

**Definition 1.3.** Let $\mathcal{F}_t \triangleq \sigma(N_u, Z_u, u \leqslant t)$ be the history of the regenerative process until time $t \in \mathbb{R}_+$. The renewal sequence $S$ is the **regeneration times** for the process $Z$, and the process $Z$ possesses the **regenerative property** of the process $(Z_{S_{n-1}+t} : t \geqslant 0)$ being independent of history $\mathcal{F}_{S_{n-1}}$ and distributed identically to $Z$.

*Remark* 1. The definition says that probability law is independent of the past and shift invariant at renewal times. That is after each renewal instant, the process becomes an independent probabilistic replica of the process starting from zero.

*Remark* 2. If the stochastic process $Z$ is bounded, then for any Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}[f(Z_t) \mid \mathcal{F}_{S_{n-1}}] = \mathbb{E}[f(Z_{t-S_{n-1}}) \mid \sigma(S_{n-1})] \mathbb{1}_{\{t \geqslant S_{n-1}\}} + f(Z_t) \mathbb{1}_{\{t < S_{n-1}\}}.$$

**Example 1.4 (Age process).** Let $N : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ be the renewal counting process for the renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, then the age at time $t$ is defined as $A_t \triangleq t - S_{N_t}$. We observe that the sample path of age in $n$th renewal interval is given by

$$A_{S_{n-1}+t} = t, \qquad t \in [0, X_n).$$

Since the segments $(X_n, (t : t \in [0, X_n)))$ are *i.i.d.* , it follows that the age process $A : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ is regenerative.

**Example 1.5 (Markov chains).** For a discrete time homogeneous, irreducible, and positive recurrent Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ on finite state space $\mathcal{X} \subset \mathbb{R}$, we can inductively define the recurrent times for state $y \in \mathcal{X}$ as $\tau_y^+(0) \triangleq 0$, and

$$\tau_y^+(n) \triangleq \inf\left\{k > \tau_y^+(n-1) : X_k = y\right\}.$$

From the strong Markov property of Markov chain $X$, it follows that $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ is a delayed renewal sequence. For all $n \in \mathbb{N}$, we define the $n$th excursion time to the state $y$ as $I_n \triangleq \left\{\tau_y^+(n-1)+1, \dots, \tau_y^+(n)\right\}$ and length of this excursion as $T_y(n) \triangleq \tau_y^+(n) - \tau_y^+(n-1)$. We can write the $n$th segment for the Markov chain $X$ as $\zeta_n = (T_y(n), (X_{\tau_y^+(n-1)+k} : k \in [T_y(n)])$. Independence of the segments follows from the strong Markov property. Further, in the segment $n \geqslant 2$ of the joint process, we can write the joint distribution for $(T_y(n), X_{\tau_y^+(n-1)+k})$ for $k < T_y(n)$ and $z \neq y$ as

$$P\left\{k < T_y(n) = m, X_{\tau_y^+(n-1)+k} = z\right\} = P_y\left\{\tau_y^+(1) > k, X_k = z\right\} P_z\left\{\tau_y^+(1) = m - k\right\}.$$

The equality follows from the strong Markov property and the homogeneity of process $X$. It follows that the Markov process $X$ is a delayed regenerative process over delayed renewal sequence $\tau_y^+$.

**Example 1.6 (Alternating renewal processes).** A renewal sequence $S:\Omega\to\mathbb{R}_+^{\mathbb{N}}$ where each inter-renewal duration $[S_{n-1},S_n)$ consists of **on** time duration $[S_{n-1},S_{n-1}+Z_n)$ followed by **off** time duration $[S_{n-1}+Z_n,S_{n-1}+Z_n+Y_n)$, is called an **alternating renewal sequence**, if $(Z,Y):\Omega\to(\mathbb{R}_+^2)^{\mathbb{N}}$ is an *i.i.d.* random sequence. The on-time duration $Z_n$ and off-time duration $Y_n$ are not necessarily independent. We denote the distributions for on, off, and renewal periods by $H,G$, and $F$, respectively. Alternating renewal processes form an important class of renewal processes, and model many interesting applications.

From the definition of $n$th inter-renewal duration $X_n\triangleq Z_n+Y_n$, we see that $X:\Omega\to\mathbb{R}_+^{\mathbb{N}}$ is an *i.i.d.* sequence, and hence $S$ is a renewal sequence. We can define an alternating stochastic process $W:\Omega\to\{0,1\}_+^{\mathbb{R}}$ that takes values 1 and 0, when the renewal process is in on and off state respectively. In particular, we can write $W_t\triangleq\mathbb{1}_{\left\{A_t\leqslant Z_{N_t+1}\right\}}$ for any time $t\in\mathbb{R}_+$.

For each $n\in\mathbb{N}$, we observe that $W_{S_{n-1}+t}=\mathbb{1}_{[0,Z_n)}(t)$ for all $t\in[0,X_n)$. Hence, we see that the $n$th segment $\zeta_n=(X_n,(\mathbb{1}_{[0,Z_n]}(t):t\in[0,X_n))$ and the segment sequence $(\zeta_n,n\in\mathbb{N})$ is *i.i.d.* , and therefore it follows that $W$ is a regenerative process over renewal sequence $S$.

**Example 1.7 (Age-dependent branching process).** Consider a population, where each organism $i$ lives for an *i.i.d.* random time period of $T_i:\Omega\to\mathbb{R}_+$ units with common distribution function $F$. Just before dying, each organism produces an *i.i.d.* random number of offsprings $N:\Omega\to\mathbb{Z}_+$, with common distribution $P$. Let $X_t$ denote the number of organisms alive at time $t$. The stochastic process $X:\Omega\to\mathbb{Z}_+^{\mathbb{R}_+}$ is called an **age-dependent branching process**. This is a popular model in biology for population growth of various organisms. We are interested in computing $m_t\triangleq\mathbb{E}X_t$ when $n=\mathbb{E}[N]=\sum_{j\in\mathbb{N}}jP_j$.

We will show that starting from an organism, the population including itself and its subsequent descendants is regenerative process. Let $T_1$ and $N_1$ denote the life time and offsprings of the first organism. If $T_1>t$, then $X_t$ is still equal to $X_0=1$. In this case, we have

$$\mathbb{E}[X_t\mathbb{1}_{\{T_1>t\}}\mid\mathcal{F}_{T_1}]=X_0\mathbb{1}_{\{T_1>t\}}. \tag{1}$$

If $T_1\leqslant t$, then $X_{T_1}=N_1$ and each of the offsprings start the population growth, independent of the past, and stochastically identical to the population growth of the original organism starting at time $T_1$. Hence, we can write $\mathbb{E}[X_t\mathbb{1}_{\{T_1\leqslant t\}}\mid\mathcal{F}_{T_1}]=\mathbb{E}[\sum_{i=0}^{N_1}X^i_{t-T_1}\mathbb{1}_{\{T_1\leqslant t\}}\mid\sigma(T_1)]$ for this case, where $(X^i_{T_1+u},u\geqslant 0)$ is a stochastic replica of $(X(u),u\geqslant 0)$, and independent for each $i\in[N_1]$. Hence for this case, we can write the following expectation conditioned on $T_1$

$$\mathbb{E}[X_t\mathbb{1}_{\{T_1\leqslant t\}}\mid\mathcal{F}_{T_1}]=\mathbb{E}[\sum_{i=1}^{N_1}X^i_{t-T_1}\mathbb{1}_{\{T_1\leqslant t\}}\mid\sigma(T_1)]=nm_{t-T_1}\mathbb{1}_{\{T_1\leqslant t\}}. \tag{2}$$

**Example 1.8 (Renewal reward process).** Consider a renewal process $S:\Omega\to\mathbb{R}_+^{\mathbb{N}}$ with *i.i.d.* inter-renewal times $X:\Omega\to\mathbb{R}_+^{\mathbb{N}}$ having common distribution $F:\mathbb{R}_+\to[0,1]$. The associated counting process is denoted by $N:\Omega\to\mathbb{Z}_+^{\mathbb{R}_+}$. We also consider an associated reward sequence $R:\Omega\to\mathbb{R}^{\mathbb{N}}$, such that a reward $R_n$ is earned at the end of the $n$th renewal interval. The reward $R_n$ can possibly depend on inter-renewal time $X_n$, but is *i.i.d.* across intervals $n\in\mathbb{N}$. That is, we assume $(X,R):\Omega\to(\mathbb{R}_+\times\mathbb{R})^{\mathbb{N}}$ be *i.i.d.* , then the **reward process** $Q:\Omega\to\mathbb{R}^{\mathbb{R}_+}$ is defined as the accumulated reward earned by time $t$ as

$$Q_t\triangleq\sum_{i=1}^{N_t}R_i.$$

The $n$th segment for process $R_{N_t+1}$ is $\zeta_n=(X_n,R_n)$. It follows that the segment sequence $\zeta:\Omega\to(\mathbb{R}_+\times\mathbb{R})^{\mathbb{N}}$ is *i.i.d.* , and hence $R_{N_t+1}$ is regenerative process with regeneration intervals being the renewal intervals $[S_{n-1},S_n)$.

## 2 Renewal equation

Let $Z : \Omega \to \mathcal{Z}^{\mathbb{R}_+}$ be a regenerative process over renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ defined on the probability space $(\Omega, \mathcal{F}, P)$, and $F$ be the distribution of inter-renewal times. The counting process associated with the renewal sequence $S$ is denoted by $N$, and we define the history of the joint process $Z, N$ until time $t$ by $\mathcal{F}_t$. For any Borel measurable set $A \in \mathcal{B}(\mathcal{Z})$ and time $t \geqslant 0$, we are interested in computing time dependent marginal probability $f_t \triangleq P\{Z_t \in A\}$ for all $t \in \mathbb{R}_+$. We can write the probability of the event $\{Z_t \in A\}$ by partitioning it into disjoint events as

$$P\{Z_t \in A\} = P\{Z_t \in A, S_1 > t\} + P\{Z_t \in A, S_1 \leqslant t\}. \tag{3}$$

We define the kernel function $K_t \triangleq P\{S_1 > t, Z_t \in A\}$ for all $t \in \mathbb{R}_+$ which is typically easy to compute for any regenerative process. By the regeneration property applied at renewal instant $S_1$, we have

$$\mathbb{E}[\mathbb{1}_{\{Z_t \in A, S_1 \leqslant t\}} \mid \mathcal{F}_{S_1}] = \mathbb{E}[\mathbb{1}_{\{Z_{t-S_1} \in A\}} \mid \sigma(S_1)]\mathbb{1}_{\{S_1 \leqslant t\}} = f_{t-S_1}\mathbb{1}_{\{S_1 \leqslant t\}}. \tag{4}$$

Taking expectation of (4) and combining with (3), we obtain the following fixed point **renewal equation** for $f$ as

$$f_t = K_t + \int_0^t dF(s) f_{t-s} = (K + F * f)_t, \quad t \in \mathbb{R}_+. \tag{5}$$

We assume that the distribution function $F$ and the kernel $K$ are known, and we wish to find $f$, and characterize its asymptotic behavior.

**Example 2.1 (Age and Excess time processes).** For a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$, we can define the age process $A : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ where the age $A_t$ at time $t$ is the time since last renewal, i.e.

$$A_t \triangleq t - S_{N_t}, \quad t \in \mathbb{R}_+.$$

Similarly, we can define the excess time process $Y : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ where the excess time $Y(t)$ at time $t$ is the time until next renewal, i.e.

$$Y_t \triangleq S_{N_t+1} - t, \quad t \in \mathbb{R}_+.$$

Since the age process is regenerative for the associated renewal sequence, we can write the renewal equation for its distribution function as

$$P\{A_t \geqslant x\} = P\{A_t \geqslant x, S_1 > t\} + \int_0^t dF(y) P\{A_{t-y} \geqslant x\}.$$

**Theorem 2.2.** *The renewal equation* (5) *has a unique solution $f = (1+m) * K$, where $m_t = \sum_{n \in \mathbb{N}} F_n(t)$ is the renewal function associated with the inter-renewal time distribution $F$.*

*Proof.* Since $F * (1+m) = m$, it follows that $K + F * (1+m) * K = (1+m) * K$, and hence $(1+m) * K$ is a solution to the renewal equation. For uniqueness, let $f$ be another solution, then $h = f - K - m * K$ satisfies $h = F * h$, and hence $h = F_n * h$ for all $n \in \mathbb{N}$. From the finiteness of $m_t$, it follows that $F_n(t) \to 0$ as $n$ grows. Hence, $\lim_{n \in \mathbb{N}} (F_n * h)_t = 0$ for each $t \in \mathbb{R}_+$. □

**Proposition 2.3.** *Let $Z$ be a regenerative process with state space $\mathcal{X} \subset \mathbb{R}$, over a renewal sequence $S$ with renewal function $m$. For a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ and the kernel function $K : \Omega \to [0,1]^{\mathbb{R}_+}$ defined as $K_t \triangleq P\{Z_t \in A, S_1 > t\}$ for all $t \in \mathbb{R}_+$, we can write*

$$P\{Z_t \in A\} = K_t + \int_0^t dm(s) K_{t-s}, \quad t \in \mathbb{R}_+.$$

**Example 2.4 (Age and excess time processes).** Since the age and excess time processes are regenerative for the associated renewal sequence, we can write the respective kernel functions $K^A, K^Y$ in the renewal equation for the respective distribution functions in terms of the complementary distribution function $\bar{F}$ of the inter-arrival times, as

$$K_t^A \triangleq P\{A_t \geqslant x, S_1 > t\} = \mathbb{1}_{\{t \geqslant x\}} \bar{F}(t), \qquad K_t^Y \triangleq P\{Y_t \geqslant x, S_1 > t\} = \bar{F}(t+x).$$

From the solution of renewal equation it follows that

$$P\{A_t \geqslant x\} = \mathbb{1}_{\{t \geqslant x\}} \bar{F}(t) + \int_0^t dm(y) \mathbb{1}_{\{t-y \geqslant x\}} \bar{F}(t-y), \quad P\{Y_t \geqslant x\} = \bar{F}(t+x) + \int_0^t dm(y) \bar{F}(t+x-y).$$

**Example 2.5 (Alternating renewal process).** Since the alternating renewal process is regenerative for the associated renewal sequence, we can write the kernel function $K : \Omega \to [0,1]^{\mathbb{R}_+}$ in the renewal equation for its distribution function in terms of the complementary distribution function $\bar{F}$ of the inter-arrival times, as $K_t \triangleq P\{W_t = 1, S_1 > t\} = P\{H_1 > t\} = \bar{H}(t)$ for all $t \in \mathbb{R}_+$. From the solution of renewal equation it follows that

$$P(t) \triangleq P\{W(t) = 1\} = \bar{H}(t) + \int_0^t dm(y) \bar{H}(t-y).$$

**Example 2.6 (Age-dependent branching process).** Combining the case of number of organisms alive before first birth $\{T_1 > t\}$ from (1), and the case of number of organisms alive after first birth $\{T_1 \leqslant t\}$ from (2), we can write the mean function $m_t$ as

$$m_t = \mathbb{E}[X_t \mathbb{1}_{\{T_1 > t\}}] + \mathbb{E}[X_t \mathbb{1}_{\{T_1 \leqslant t\}}] = \bar{F}(t) + n \int_0^t m_{t-u} dF(u). \tag{6}$$

This looks almost like a renewal function. Multiplying both sides of the above equation by $e^{-\alpha t}$, we get

$$m_t e^{-\alpha t} = e^{-\alpha t} \bar{F}(t) + n \int_0^t e^{-\alpha(t-u)} m_{t-u} e^{-\alpha u} dF(u).$$

We define $dG(t) \triangleq n e^{-\alpha t} dF(t)$, then the following choice of $\alpha > 0$ ensures that $G : \mathbb{R}_+ \to [0,1]$ is a distribution function on $\mathbb{R}_+$. In particular, $\alpha$ is chosen to be the unique solution to the equation

$$1 = n \int_0^\infty e^{-\alpha t} dF(t).$$

With this choice of distribution function $G$, the above equation (6) is a renewal equation for the function $f : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ defined as $f_t \triangleq e^{-\alpha t} m_t$ for all $t \in \mathbb{R}_+$ with the kernel function $K : \Omega \to [0,1]^{\mathbb{R}_+}$ defined as $K_t \triangleq e^{-\alpha t} \bar{F}(t)$ for all $t \in \mathbb{R}_+$ and the common inter-renewal time distribution being $G$. That is,

$$f = K + f * G.$$

Define the inter-renewal time distribution $G_1 \triangleq G$ to inductively define the $n$th renewal time distribution $G_n \triangleq G_{n-1} * G$ and the associated renewal function $m^G \triangleq \sum_{n \in \mathbb{N}} G_n$, to write the solution of the renewal equation (6) as

$$m_t e^{-\alpha t} = e^{-\alpha t} \bar{F}(t) + \int_0^t e^{-\alpha(t-u)} \bar{F}(t-u) dm_u^G.$$

**Example 2.7 (Renewal Reward Process).** Considering the event of no renewal or at least one renewal before time $t$ for the regenerative process $R_{N_t+1}$, we can write

$$g_t \triangleq \mathbb{E}[R_{N_t+1}] = \mathbb{E}[R_{N_t+1} \mathbb{1}_{\{S_1 > t\}}] + \mathbb{E}[R_{N_t+1} \mathbb{1}_{\{S_1 \leqslant t\}}]$$

From the definition of counting process $N$, we have $t \mapsto K_t \triangleq \mathbb{E}[R_{N_t+1} \mathbb{1}_{\{S_1 > t\}}] = \mathbb{E}[R_1 \mathbb{1}_{\{X_1 > t\}}]$. Further, from the regenerative property of $R_{N_t+1}$, we obtain $\mathbb{E}[R_{N_t+1} \mathbb{1}_{\{S_1 \leqslant t\}} \mid \mathcal{F}_{S_1}] = \mathbb{1}_{\{S_1 \leqslant t\}} g_{t-S_1}$. Thus, $\mathbb{E}[R_{N_t+1} \mathbb{1}_{\{S_1 \leqslant t\}}] = \mathbb{E}[\mathbb{1}_{\{S_1 \leqslant t\}} g_{t-S_1}]$. Combining the two case, we get the renewal equation $g = K + g * F$, which has the unique solution $g = K + K * m$.

## 3 Inspection Paradox

**Lemma 3.1 (Inspection Paradox).** *For a renewal process* $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *with inter-arrival times* $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *and associated counting process* $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$*, we have* $\mathbb{E}X_{N_t+1} \geqslant \mathbb{E}X_1$.

*Proof.* It suffices to show that $t \mapsto g_t^x \triangleq P\{X_{N_t+1} > x\} \geqslant \bar{F}(x)$ for all $x, t \in \mathbb{R}_+$. To this end, we first recall that $X_{N_t+1}$ is a regenerative process with regeneration instant sequence $S$ since its segment during the $n$th renewal period $[S_{n-1}, S_n)$ is $\xi_n = (X_n, (X_n, t \in [S_{n-1}, S_n)))$. Defining the kernel function $t \mapsto k_t^x \triangleq P\{X_{N_t+1} > x, S_1 > t\} = \bar{F}(x \vee t)$, we can write the solution to the renewal equation as $g^x = k^x * (1+m)$.

From the Chebyshev's inequality for the increasing functions $z \mapsto f(z) \triangleq \mathbb{1}_{\{z>x\}}$ and $z \mapsto g(z) \triangleq \mathbb{1}_{\{z>t\}}$ and random variable $X_1$, we can write $k_t = \mathbb{E}\mathbb{1}_{\{X_{N_t+1}>x, X_1>t\}} = \mathbb{E}\mathbb{1}_{\{X_1>x, X_1>t\}} \geqslant \bar{F}(x)\bar{F}(t)$. Since $\bar{F} * (1+m) = 1$, it follows that $g = k * (1+m) \geqslant \bar{F}(x)$ and hence

$$\mathbb{E}X_{N_t+1} = \int_{x \in \mathbb{R}_+} g_t dx \geqslant \int_{x \in \mathbb{R}_+} \bar{F}(x) dx = \mathbb{E}X_1.$$

Alternatively, we observe that $\bar{F}(x \vee t) \geqslant \bar{F}(x)\bar{F}(t)$ and hence the result follow. □

*Remark* 3. The accumulated reward $R_{N_t+1}$ in the current renewal interval is possibly dependent on the current renewal duration $X_{N_t+1}$. If the reward accrual rate is positive, then it follows from the inspection paradox that $\mathbb{E}R_{N_t+1} \geqslant \mathbb{E}R_1$.

**Lemma 3.2.** *For a renewal reward process with positive reward accrual rate, we have* $\mathbb{E}R_{N_t+1} \geqslant \mathbb{E}R_1$.

*Proof.* Recall that $R_{N_t+1}$ is a regenerative process, and we can write the solution to the renewal equation for its tail probability $t \mapsto f_t^x \triangleq P\{R_{N_t+1} > x\}$ in terms of the kernel function $t \mapsto k_t^x \triangleq P\{R_{N_t+1} > x, S_1 > t\}$ as $f^x = k^x * (1+m)$. From the distribution functions $F, H$ for $X_1, R_1$ and Chebyshev's inequality applied to increasing indicator functions of random variable $X_1$, we obtain $k_t^x \geqslant \bar{H}(x)\bar{F}(t)$. Since $\bar{F} * (1+m) = 1$, it follows that $f^x \geqslant \bar{H}(x)$ and hence $\mathbb{E}R_{N_t+1} \geqslant \mathbb{E}R_1$. □

## A Chebyshev's sum inequality

**Theorem A.1.** *Consider two non-decreasing positive measurable functions* $f, g : \mathbb{R} \to \mathbb{R}_+$ *and a random variable* $X : \Omega \to \mathbb{R}$*. Then,* $\mathbb{E}f(X)g(X) \geqslant \mathbb{E}f(X)\mathbb{E}g(X)$.

*Proof.* Consider a random vector $Y : \Omega \to \mathbb{R}^2$ to be an *i.i.d.* replica of $X : \Omega \to \mathbb{R}$ and the product $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))$. From the linearity of expectation and $Y_1, Y_2$ being *i.i.d.* to $X$, we can expand the mean of the product as

$$\mathbb{E}(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2)) = 2\mathbb{E}f(X)g(X) - 2\mathbb{E}f(X)\mathbb{E}g(X) = 2\mathbb{E}(f(X) - \mathbb{E}f(X))(g(X) - \mathbb{E}g(X)).$$

Since $f, g$ are non-decreasing, we have $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))\mathbb{1}_{\{Y_1 \geqslant Y_2\}} \geqslant 0$ and $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))\mathbb{1}_{\{Y_1 < Y_2\}} \geqslant 0$. Summing the two terms, we obtain

$$(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2)) \geqslant 0.$$

□