Lecture-11: Key Lemma and Blackwell Theorem

1 Key Lemma

Theorem 1.1 (Key Lemma). *Consider a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with* i.i.d. *inter-renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having common distribution function F, associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$, and the renewal function $m : \mathbb{R}_+ \to \mathbb{R}_+$. Then,*

$$P\{S_{N_t} \leqslant s\} = \bar{F}(t) + \int_0^s \bar{F}(t-y)dm(y), \qquad t \geqslant s \geqslant 0.$$

Proof. We can see that event of time of last renewal prior to t being smaller than another time s can be partitioned into disjoint events corresponding to number of renewals until time t. Each of these disjoint events is equivalent to occurrence of nth renewal before time s and $(n+1)$th renewal past time t. That is,

$$\{S_{N_t} \leqslant s\} = \bigcup_{n \in \mathbb{Z}_+} \{S_{N_t} \leqslant s, N_t = n\} = \bigcup_{n \in \mathbb{Z}_+} \{S_n \leqslant s, S_{n+1} > t\}.$$

Recognizing that $S_0 = 0, S_1 = X_1$, and that $S_{n+1} = S_n + X_{n+1}$, we can write

$$P\{S_{N_t} \leqslant s\} = P\{X_1 > t\} + \sum_{n \in \mathbb{N}} \mathbb{E}[\mathbb{1}_{\{S_n \leqslant s\}} \mathbb{E}[\mathbb{1}_{\{X_{n+1} > t - S_n\}} | \sigma(S_n)]].$$

We recall that the distribution function of nth renewal instant S_n is the n-fold convolution of F denoted by F_n. Taking expectation of $\bar{F}(t - S_n)\mathbb{1}_{\{S_n \leqslant s\}}$, we get

$$P\{S_{N_t} \leqslant s\} = \bar{F}(t) + \sum_{n \in \mathbb{N}} \int_{y=0}^{s} \bar{F}(t-y)dF_n(y).$$

Using monotone convergence theorem to interchange integral and summation, and noticing that $m(y) = \sum_{n \in \mathbb{N}} F_n(y)$, the result follows. □

> **Exercise 1.2.** Prove the key lemma using the marginal distribution of age process, and the fact that $A_t = t - S_{N_t}$ for all $t \in \mathbb{R}_+$. Recall that the age process is regenerative, and hence its marginal distribution can be obtained as the solution to its corresponding renewal equation.

Remark 1. Key lemma tells us that distribution of S_{N_t} has probability mass at 0 and density between $(0, t]$, that is,

$$P\{S_{N_t} = 0\} = \bar{F}(t), \qquad dF_{S_{N_t}}(y) = \bar{F}(t-y)dm(y), \quad 0 < y \leqslant t.$$

Remark 2. Probability of nth renewal taking place in the duration $[y, y+dy]$ is given by $P\{S_n \in (y, y+dy)\} = dF_n(y)$. Therefore, probability of some renewal taking place in the infinitesimal neighborhood of y, is

$$P\Big(\cup_{n \in \mathbb{N}} \{S_n \in (y, y+dy)\}\Big) = \sum_{n \in \mathbb{N}} dF_n(y) = dm(y).$$

Probability of no renewal in the interval $(y+dy, t]$, given the nth renewal occurred at time y, is given by $P\{X_{n+1} > t - y\} = \bar{F}(t-y)$. It follows that

$$P\{\text{renewal occurs in } (y, y+dy) \text{ and next arrival after } t-y\} = dF_{S_{N_t}}(y).$$

That is, the density of last renewal time S_{N_t} has the interpretation of renewal taking place in the infinitesimal neighborhood of y, and no renewal in the duration $[y, t]$.

Example 1.3 (Poisson process). Let the inter-renewal time *i.i.d.* random sequence $X : \Omega : \mathbb{R}_+^{\mathbb{N}}$ be exponentially distributed with common distribution $F(x) = 1 - e^{-\lambda x}$ for $x \in \mathbb{R}_+$. Then, the distribution of last renewal is given by

$$P\{S_{N_t} \leqslant x\} = e^{-\lambda t} + \int_0^x \lambda e^{-\lambda(t-y)} dy = e^{-\lambda(t-x)}, \qquad 0 \leqslant x \leqslant t.$$

Exercise 1.4. Find the age and the excess time distribution for a Poisson process.

2 Delayed Regenerative Process

We are ready to study the delayed regenerative processes $Z : \Omega \to \mathbb{R}^{\mathbb{R}_+}$, with delayed regeneration points $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and counting process $N_D : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. We assume that inter-renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is independent with distributions $G \triangleq F_{X_1}$ and $F \triangleq F_{X_n}$ for all $n \geqslant 2$. That is, $(X_n : n \geqslant 2)$ is *i.i.d.* . We denote the distribution of nth renewal instant S_n by $G * F_{n-1}$ where F_n is n-fold convolution of F, and the delayed renewal function by $m_D = \sum_{n\in\mathbb{N}} G * F_{n-1}$. The nth segment of the joint process (N_D, Z) is given by $\zeta_n \triangleq (X_n, (Z_t, t \in [S_{n-1}, S_n) : n \in \mathbb{N})$, and in the delayed regenerative process, the segments ζ are independent and $(\zeta_n : n \geqslant 2)$ is *i.i.d.* . In this case, $(Z_{S_n+t} : t \in [0, X_{n+1}))$ is independent of $\mathcal{F}_{S_n}$ and distributed identically to $(Z_{S_1+t}, t \in [0, X_2))$ for all $n \in \mathbb{N}$.

Theorem 2.1. *For a delayed regenerative process $Z : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ and a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$, we define the probability function $f : \mathbb{R}_+ \to [0,1]$ as $f_t \triangleq P\{Z_t \in A\}$, the kernel functions $K_1, K_2 : \mathbb{R}_+ \to [0,1]$ such that $K_1(t) \triangleq P\{Z_t \in A, S_1 > t\}, K_2(t) \triangleq P\{Z_{S_1+t} \in A, t \in [0, X_2)\}$ for all $t \in \mathbb{R}_+$. Then, $f = K_1 + K_2 * m_D$.*

Proof. For a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$, we can write the probability $f_t = P\{Z_t \in A\}$ of the delayed regenerative process taking values in this set A as disjoint sum of probability of disjoint partitions of this event $\{Z_t \in A, N_t = n\}$ as

$$f_t = P\{Z_t \in A, S_1 > t\} + \sum_{n\in\mathbb{N}} \mathbb{E}[\mathbb{1}_{\{Z_t\in A\}} \mathbb{1}_{[S_n,S_{n+1})}(t)].$$

Using the tower property of conditional expectation and the regenerative property of Z, we can write

$$\mathbb{E}[\mathbb{1}_{\{Z_t\in A\}}\mathbb{1}_{[S_n,S_{n+1})}(t)] = \mathbb{E}[\mathbb{1}_{\{S_n\leqslant t\}}\mathbb{E}[\mathbb{1}_{\{Z_{S_n+t-S_n}\in A\}}\mathbb{1}_{[0,X_{n+1})}(t-S_n) \mid \mathcal{F}_{S_n}]] = \mathbb{E}[\mathbb{1}_{\{S_n\leqslant t\}}K_2(t-S_n)].$$

The result follows from aggregating the results for all $n \in \mathbb{N}$, and the fact that $m_D = \sum_{n\in\mathbb{N}} F_{S_n}$. ☐

Example 2.2 (Age process). For a delayed renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, consider the age process $A : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ defined as $A_t \triangleq t - S_{N_t}$ for all $t \in \mathbb{R}_+$, and its nth segment given by $\zeta_n \triangleq (X_n, (A_{S_{n-1}+t} : t \in [0, X_n))) = (X_n, (t : t \in [0, X_n)))$. Since segments are independent, and identically distributed for $n \geqslant 2$, it follows that A is a delayed regenerative process. For a measurable set $B \triangleq [x, \infty)$, then we can compute the kernel functions

$$K_1(t) \triangleq P\{A_t \geqslant x, S_1 > t\} = 1_{\{t\geqslant x\}}\bar{G}(t), \qquad K_2(t) \triangleq P\{A_{S_1+t} \geqslant x, S_1 + t \in [0, X_2)\} = 1_{\{t\geqslant x\}}\bar{F}(t).$$

Therefore, we can write the distribution of last renewal time for the delayed renewal process as

$$P\{S_{N_t} \leqslant x\} = P\{A_t \geqslant t - x\} = \mathbb{1}_{\{x\geqslant 0\}}\bar{G}(t) + \int_0^t dm^D(y)\mathbb{1}_{\{t-y\geqslant t-x\}}\bar{F}(t-y).$$

Corollary 2.3 (Delayed Key Lemma). *Consider a delayed renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with independent inter-renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with first renewal time having distribution G and common distribution F*

for inter-renewal times $(X_n, n \geqslant 2)$, *associated counting process* $N^D : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$, *and the renewal function* $m_D : \mathbb{R}_+ \to \mathbb{R}_+$. *Then,*

$$P\left\{S_{N_t^D} \leqslant s\right\} = \bar{G}(t) + \int_0^s \bar{F}(t-y)dm_D(y), \qquad t \geqslant s \geqslant 0.$$

3 Blackwell Theorem

Lemma 3.1. *For a renewal sequence* S, *let* F *be the inter-renewal time distribution such that* $\inf\{x \in \mathbb{R}_+ : F(x) = 1\} = \infty$, *then for any* $b > 0$

$$\sup\{m_t - m_{t-b} : t \in \mathbb{R}_+\} < \infty.$$

Proof. Recall that $m = \sum_{n\in\mathbb{N}} F_n$ and hence $m * F = m - F$. This implies that $m * (1-F) = F$. Since the function $1 - F$ is monotonically non-increasing, $\inf_{s\in[0,b]} \bar{F}(s) = \bar{F}(b)$. Therefore,

$$1 \geqslant F(t) = \int_0^t dm(s)\bar{F}(t-s) \geqslant \int_{t-b}^t dm(s)\bar{F}(t-s) \geqslant [m_t - m_{t-b}]\bar{F}(b),$$

where b is chosen so that $F(b) < 1$. Hence, the result follows. □

Theorem 3.2 (Blackwell's Theorem). *Consider a renewal sequence* S *with the inter-renewal time distribution* F *such that* $\inf\{x \in \mathbb{R}_+ : F(x) = 1\} = \infty$, *mean of inter-renewal time* μ, *and renewal function* m. *If* F *is not lattice, then for all* $a \geqslant 0$

$$\lim_{t\to\infty} m_{t+a} - m_t = \frac{a}{\mu}.$$

If F *is lattice with period* d, *then* $\lim_{n\to\infty} m_{(n+1)d} - m_{nd} = \frac{d}{\mu}$.

Proof. We will not prove that the following limit exists for non-lattice F,

$$g(a) \triangleq \lim_{t\to\infty}[m_{t+a} - m_t] \tag{1}$$

However, we show that if this limit does exist, it is equal to $\frac{a}{\mu}$ as a consequence of elementary renewal theorem. To this end, note that $m_{t+a+b} - m_t = m_{t+a+b} - m_{t+a} + m_{t+a} - m_t$. Taking limits on both sides of the above equation, we conclude that $g(a+b) = g(a) + g(b)$. The only increasing solution of such a $g : \mathbb{R}_+ \to \mathbb{R}_+$ for all $a > 0$ is $g(a) = ca$, for some positive constant c. To show $c = \frac{1}{\mu}$, define a sequence $x \in \mathbb{R}_+^{\mathbb{N}}$ in terms of renewal function m_t, as

$$x_n \triangleq m_n - m_{n-1},\ n \in \mathbb{N}.$$

Note that $\sum_{i=1}^n x_i = m_n$ and $\lim_{n\in\mathbb{N}} x_n = g(1) = c$. Further recall that, if a sequence $x \in \mathbb{R}^{\mathbb{N}}$ converges, then the running average sequence $a \in \mathbb{R}^{\mathbb{N}}$ defined as $a_n \triangleq \frac{1}{n}\sum_{i=1}^n x_i$ converges to the same limit. Hence, we have the Cesàro mean converging to $\lim_{n\in\mathbb{N}} \frac{\sum_{i=1}^n x_i}{n} = \lim_{n\in\mathbb{N}} \frac{m_n}{n} = c$. Therefore, we can conclude $c = \frac{1}{\mu}$ by elementary renewal theorem.

When F is lattice with period d, the limit in (1) doesn't exist. (See the following example). However, the theorem is true for lattice again by elementary renewal theorem. Indeed, since $\frac{m_{nd}}{n} \to \frac{1}{\mu}$, we can define $x_n \triangleq m_{nd} - m_{(n-1)d}$ and observe that $\sum_{i=1}^n x_i = m_{nd}$ and the Cesàro mean $\frac{1}{n}\sum_{i=1}^n x_i$ converges to $\frac{d}{\mu}$ by elementary renewal theorem. □

Example 3.3. Consider a renewal process with $P\{X_n = 1\} = 1$, that is, there is a renewal at every positive integer time instant with unit probability. Then F is lattice with $d = 1$. Now, for $a = 0.25$, and $t_n = n + (-1)^n a$, we see that $m_{t_n} = N_{t_n} = n - \mathbb{1}_{\{n \text{ odd}\}}$, and $m_{t_n+a} = n$. It follows that $m_{t_n+a} - m_{t_n} = \mathbb{1}_{\{n \text{ odd}\}}$, and hence $\lim_{t_n\to\infty} m_{t_n+a} - m_{t_n}$ does not exist. It follows that $\lim_{t\to\infty} m_{t+a} - m_t$ does not exist.

Exercise 3.4. Let m be the renewal function associated with a non-lattice F. Show that the following limit exists

$$g(a) \triangleq \lim_{t\to\infty}[m_{t+a} - m_t].$$

Remark 3. In the lattice case, if the inter arrivals are strictly positive, that is, there can be no more than one renewal at each nd, then we have that

$$\lim_{n\to\infty} P\{\text{renewal at } nd\} = \frac{d}{\mu}.$$

Corollary 3.5 (Delayed Blackwell's Theorem). *Consider a delayed renewal process with independent inter-renewal times, with the distribution of first renewal being G with mean μ_G, and the distribution of inter-renewal times for $n \geqslant 2$ being F with mean μ_F and the property* $\inf\{x : F(x) = 1\} = \infty$. *Let the associated renewal function be m^D and F is not lattice, then for all $a \geqslant 0$*

$$\lim_{t\to\infty} m^D_{t+a} - m^D_t = \frac{a}{\mu_F}.$$

If F and G are lattice with period d, then $\lim_{n\to\infty} m^D_{(n+1)d} - m^D_{nd} = \frac{d}{\mu_F}$.