Lecture-12: Key Renewal Theorem

1 Key Renewal Theorem

Theorem 1.1 (Key renewal theorem). *Consider a recurrent renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with renewal function $m : \mathbb{R}_+ \to \mathbb{R}_+$, the common mean μ, and the distributionF for* i.i.d. *inter-renewal times. For any directly Riemann integrable function $z \in \mathbb{D}$, we have*

$$\lim_{t\to\infty} \int_0^t z(t-x)dm(x) = \begin{cases} \frac{1}{\mu}\int_0^\infty z(t)dt, & F \text{ is non-lattice,} \\ \frac{d}{\mu}\sum_{k\in\mathbb{Z}_+} z(t+kd), & F \text{ is lattice with period } d, \quad t = nd. \end{cases}$$

Proposition 1.2 (Equivalence). *Blackwell's theorem and key renewal theorem are equivalent.*

Proof. Let's assume key renewal theorem is true. We select $z : \mathbb{R}_+ \to \mathbb{R}_+$ as a simple function with value unity on interval $[0, a]$ for $a \geqslant 0$ and zero elsewhere. That is, $z(t) = \mathbb{1}_{[0,a]}(t)$ for any $t \in \mathbb{R}_+$. From Proposition A.3, it follows that z is directly Riemann integrable. Therefore, by Key Renewal Theorem, we have

$$\lim_{t\to\infty}[m(t) - m(t-a)] = \frac{a}{\mu}.$$

We defer the formal proof of converse for a later stage. We observe that, from Blackwell theorem, it follows

$$\lim_{t\to\infty}\frac{dm(t)}{dt} \overset{(a)}{=} \lim_{a\to 0}\lim_{t\to\infty}\frac{m(t+a)-m(t)}{a} = \frac{1}{\mu}.$$

where in (a) we can exchange the order of limits under certain regularity conditions. □

Remark 1. Key renewal theorem is very useful in computing the limiting value of some function g, where g_t is a probability or expectation of an event at an arbitrary time t, for a regenerative process. This value is computed by conditioning on the time of last regeneration prior to time t.

Corollary 1.3 (Delayed key renewal theorem). *Consider an aperiodic and recurrent delayed renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with independent inter-arrival times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with first inter-renewal time distribution G and common inter-renewal time distribution F for $(X_n : n \geqslant 2)$. Let the renewal function be denoted by m_D and means $\mathbb{E}X_1 = \mu_G$ and $\mathbb{E}X_2 = \mu_F$. For any directly Riemann integrable function $z \in \mathbb{D}$ and F non-lattice, we have*

$$\lim_{t\to\infty}\int_0^t z(t-x)dm_D(x) = \frac{1}{\mu_F}\int_0^\infty z(t)dt.$$

Remark 2. Any kernel function $t \mapsto K_t \triangleq P\{Z_t \in A, X_1 > t\} \leqslant \bar{F}(t)$, and hence is d.R.i. from Proposition A.3(b).

Example 1.4 (Limiting distribution of regenerative process). For a regenerative process Z over a delayed renewal process S with finite mean *i.i.d.* inter-arrival times, we have $K_2(t) \triangleq P\{Z_{S_1+t} \in A, X_2 > t\} \leqslant \bar{F}(t)$ for any $A \in \mathcal{B}(\mathbb{R})$, and hence the kernel function $K_2 \in \mathbb{D}$. Applying Key Renewal Theorem to renewal function, we get the limiting probability of the event $\{Z_t \in A\}$ as

$$\lim_{t\to\infty} P\{Z_t \in A\} = \lim_{t\to\infty}(m_D * K_2)(t) = \frac{1}{\mu_F}\int_{t=0}^\infty K_2(t)dt.$$

Example 1.5 (Limiting distribution of age and excess time). For a delayed renewal process S with finite mean independent inter-renewal times such that the distribution of first renewal time is G, and the distribution of subsequent renewal times are identically F. Denoting the associated counting process by N_D and renewal function m_D, we can write the limiting probability distribution of age as $F_e(x) \triangleq \lim_{t\to\infty} P\{A_t \leqslant x\}$. We can write the complementary distribution as

$$\bar{F}_e(x) = \lim_{t\to\infty} P\{A_t \geqslant x\} = \lim_{t\to\infty} \int_0^t dm_D(t-y)\mathbb{1}_{\{y\geqslant x\}}\bar{F}(y) = \frac{1}{\mu_F}\int_x^\infty \bar{F}(y)dy.$$

Example 1.6 (Limiting on probability of alternating renewal process). Consider an alternating renewal process W with random on and off time sequence Z and Y respectively, such that (Z,Y) is *i.i.d.* . We denote the distribution of on and off times by non-lattice functions H and G respectively. If $\mathbb{E}Z_n$ and $\mathbb{E}Y_n$ are finite, then applying Key renewal theorem to the limiting probability of alternating process being on, we get

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}(m * \bar{H})(t) = \frac{\mathbb{E}Z_n}{\mathbb{E}Z_n + \mathbb{E}Y_n}.$$

Example 1.7 (Limiting population in age dependent branching processes). For a population evolution as age dependent branching process with initial population of unity, and *i.i.d.* lifetimes with common distribution F and the mean number of progenies n, the mean number of organisms alive at time t is

$$m_t e^{-\alpha t} = e^{-\alpha t}\bar{F}(t) + \int_0^t e^{-\alpha(t-u)}\bar{F}(t-u)dm_u^G,$$

where α is chosen to be the unique solution to the equation $1 = n\int_0^\infty e^{-\alpha t}dF(t)$, and m^G is the renewal function associated with the distribution G defined as $dG(t) \triangleq ne^{-\alpha t}dF(t)$. Since the kernel function $e^{-\alpha t}\bar{F}(t)$ is non-negative, monotone non-increasing, and integrable, it is directly Riemann integrable. Hence, we can apply key renewal theorem to obtain the following limit

$$\lim_{t\to\infty} m_t e^{-\alpha t} = \frac{1}{\mu_G}\int_0^\infty e^{-\alpha t}\bar{F}(t)dt = \frac{\int_0^\infty e^{-\alpha t}\bar{F}(t)dt}{n\int_0^\infty te^{-\alpha t}dF(t)}.$$

Using integration by parts, we can write the numerator as $\int_0^\infty e^{-\alpha t}\bar{F}(t)dt = \frac{1}{\alpha} - \frac{1}{\alpha}\int_0^\infty e^{-\alpha t}dF(t) = \frac{1}{\alpha}\left(1 - \frac{1}{n}\right)$.

Exercise 1.8 (Age and excess time process as an alternating renewal process). Consider a renewal sequence with a non-lattice distribution F for *i.i.d.* inter-renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ such that $\mathbb{E}X_1^2 < \infty$. For each $x \in \mathbb{R}_+$, we can define an alternating renewal process $W : \Omega \to [0,1]^{\mathbb{R}_+}$ defined as $W_t \triangleq \mathbb{1}_{\{A_t \leqslant x\}}$.

(a) Show that W is a regenerative alternating process.
(b) Show that its nth on and off times are $Z_n \triangleq X_n \wedge x$ and $Y_n \triangleq X_n - Z_n$ respectively.
(c) Repeat the same exercise when on times are excess time being less than a threshold x.
(d) Show that the limiting age and excess time distributions are identical to F_e.
(e) Show that the limiting mean of age and excess times satisfy the following equality,

$$\lim_{t\to\infty} \mathbb{E}Y_t = \lim_{t\to\infty} \mathbb{E}A_t = \frac{\mathbb{E}X_1^2}{2\mathbb{E}X_1}.$$

(f) Show that $\lim_{t\to\infty}\left(m_t - \frac{t}{\mathbb{E}X_1}\right) = \frac{\mathbb{E}X_1^2}{2(\mathbb{E}X_1)^2} - 1$.

A Directly Riemann Integrable

For each scalar $h > 0$ and natural number $n \in \mathbb{N}$, we can define intervals $I_n(h) \triangleq [(n-1)h, nh)$, such that the collection $(I_n(h), n \in \mathbb{N})$ partitions the positive real-line $\mathbb{R}_+$. For any function $z : \mathbb{R}_+ \to \mathbb{R}_+$ be a function bounded over finite intervals, we can denote the infimum and supremum of z in the interval I_n as

$$\underline{z}_h(n) \triangleq \inf\{z(t) : t \in I_n(h)\} \qquad \overline{z}_h(n) \triangleq \sup\{z(t) : t \in I_n(h)\}.$$

We can define functions $\underline{z}_h, \overline{z}_h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\underline{z}_h(t) \triangleq \sum_{n\in\mathbb{N}} \underline{z}_h(n)\mathbb{1}_{I_n(h)}(t)$ and $\overline{z}_h(t) \triangleq \sum_{n\in\mathbb{N}} \overline{z}_h(n)\mathbb{1}_{I_n(h)}(t)$ for all $t \in \mathbb{R}_+$. From the definition, we have $\underline{z}_h \leqslant z \leqslant \overline{z}_h$ for all $h \geqslant 0$. The infinite sums of infimum and supremums over all the intervals $(I_n(h), n \in \mathbb{N})$ are denoted by

$$\int_{t\in\mathbb{R}_+} \underline{z}_h(t)dt = h \sum_{n\in\mathbb{N}} \underline{z}_h(n), \qquad \int_{t\in\mathbb{R}_+} \overline{z}_h(t)dt = h \sum_{n\in\mathbb{N}} \overline{z}_h(n).$$

Remark 3. Since $\underline{z}_h \leqslant z \leqslant \overline{z}_h$, we observe that $\int_{t\in\mathbb{R}_+} \underline{z}_h(t)dt \leqslant \int_{t\in\mathbb{R}_+} \overline{z}_h(t)dt$. We observe that $\underline{z}_h$ and $\overline{z}_h$ are non-decreasing and non-increasing in h respectively. As as $h \downarrow 0$, if both left and right limits exist and are equal, then the integral value $\int_{t\in\mathbb{R}_+} z(t)dt$ is equal to the limit.

Definition A.1 (directly Riemann integrable (d.R.i.)). A function $z : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is **directly Riemann integrable** and denoted by $z \in \mathbb{D}$ if the partial sums obtained by summing the infimum and supremum of h, taken over intervals obtained by partitioning the positive axis, are finite and both converge to the same limit, for all finite positive interval lengths. That is,

$$\sum_{n\in\mathbb{N}} h\overline{z}_h(n) < \infty, \qquad \lim_{h\downarrow 0} \int_{t\in\mathbb{R}_+} \overline{z}_h(t)dt = \lim_{h\downarrow 0} \int_{t\in\mathbb{R}_+} \underline{z}_h(t)dt.$$

The limit is denoted by $\int_{t\in\mathbb{R}_+} z(t)dt = \lim_{h\downarrow 0} \sum_{n\in\mathbb{N}} h\overline{z}_h(n) = \lim_{h\downarrow 0} \sum_{n\in\mathbb{N}} h\underline{z}_h(n)$. For a real function $z : \mathbb{R}_+ \to \mathbb{R}$, we can define the positive and negative parts by $z^+, z^- : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$ $z^+(t) \triangleq z(t) \vee 0$, and $z^-(t) \triangleq -(z(t) \wedge 0)$. If both $z^+, z^- \in \mathbb{D}$, then $z \in \mathbb{D}$ and the limit is

$$\int_{\mathbb{R}_+} z(t)dt \triangleq \int_{\mathbb{R}_+} z^+(t)dt - \int_{\mathbb{R}_+} z^-(t)dt.$$

Remark 4. We compare the definitions of directly Riemann integrable and Riemann integrable functions. For a finite positive M, a function $z : [0, M] \to \mathbb{R}$ is Riemann integrable if

$$\lim_{h\to 0} \int_0^M \underline{z}_h(t)dt = \lim_{h\to 0} h \int_0^M \underline{z}_h(t)dt.$$

In this case, the limit is the value of the integral $\int_0^M z(t)dt$. For a function $z : \mathbb{R}_+ \to \mathbb{R}$,

$$\int_{t\in\mathbb{R}_+} z(t)dt = \lim_{M\to\infty} \int_0^M z(t)dt,$$

if the limit exists. For many functions, this limit may not exist.

Remark 5. A directly Riemann integrable function over $\mathbb{R}_+$ is also Riemann integrable, but the converse need not be true. For instance, we can define $E_n \triangleq \left[n - \frac{1}{2n^2}, n + \frac{1}{2n^2}\right]$ for each $n \in \mathbb{N}$, and consider the following Riemann integrable function $z : \mathbb{R}_+ \to \mathbb{R}_+$

$$z(t) = \sum_{n\in\mathbb{N}} \mathbb{1}_{E_n}(t), \qquad t \in \mathbb{R}_+.$$

We observe that z is Riemann integrable, however $\int_{t\in\mathbb{R}_+} \overline{z}(t)dt$ is always infinite. It suffices to show that $h\sum_{n\in\mathbb{N}} \overline{z}_h(n)$ is always infinite for every $h > 0$. Since the collection $(I_n(h) : n \in N)$ partitions the entire $\mathbb{R}_+$, for each $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that $E_n \cap I_m(h) \neq \emptyset$, and therefore $\overline{z}_m(h) = 1$. It follows that

$$\int_{t\in\mathbb{R}_+} \overline{z}(t)dt = \sum_{m\in\mathbb{N}} h = \infty.$$

Exercise A.2 (Necessary conditions for d.R.i.). If a function $z : \mathbb{R}_+ \to \mathbb{R}_+$ is directly Riemann integrable, then show that z is bounded and continuous a.e.

Exercise A.3 (Sufficient conditions for d.R.i.). Show that if any of the following conditions hold for a function $z : \mathbb{R}_+ \to \mathbb{R}_+$, then it is directly Riemann integrable.

(a) z is monotone non-increasing, and Lebesgue integrable.

(b) z is bounded above by a directly Riemann integrable function.

(c) z has bounded support.

(d) z is continuous, and has finite support.

(e) z is continuous, bounded, and $\overline{\sigma}_\delta$ is bounded for some δ.

(f) $\int_{t\in\mathbb{R}_+} \overline{z}_h(t)dt$ is bounded for some $h > 0$.

Exercise A.4. For any directly Riemann integrable function $z : \mathbb{R}_+ \to \mathbb{R}_+$ show that $\lim_{t\to\infty} z(t) = \lim_{n\to\infty} \overline{z}_h(n)$.

Proposition A.5 (Tail Property). *If $z : \mathbb{R}_+ \to \mathbb{R}_+$ is directly Riemann integrable and has bounded integral value, then $\lim_{t\to\infty} z(t) = 0$.*

Proof. If $z \in \mathbb{D}$, then $h\sum_{n\in\mathbb{N}} \overline{z}_h(n) < \infty$ for all $h > 0$. This implies that the infinite positive sum $\sum_n \overline{z}_h(n)$ is finite, and hence $\lim_{n\to\infty} \overline{z}_h(n) = \lim_{t\to\infty} z(t) = 0$. □

Corollary A.6. *Any distribution $F : \mathbb{R}_+ \to [0,1]$ with finite mean μ, the complementary distribution function $\bar{F}$ is d.R.i.*

Proof. Since $\bar{F}$ is monotonically non-increasing and its Lebesgue integration is $\int_{\mathbb{R}_+} \bar{F}(t)dt = \mu$, the result follows from Proposition A.3(a). □