Lecture-13: Applications of Key Renewal Theorem

1 Equilibrium renewal process

Definition 1.1. The limiting distribution of age for a renewal process with common inter-renewal duration distribution $F : \mathbb{R}_+ \to [0,1]$ is given by the **equilibrium distribution** $F_e : \mathbb{R}_+ \to [0,1]$ defined as $F_e(x) \triangleq \frac{1}{\mu_F}\int_0^x \bar{F}(y)dy$ for all $x \in \mathbb{R}_+$.

Lemma 1.2. *The moment generating function of $F_e(x)$ is $\tilde{F}_e(s) = \frac{1-\tilde{F}(s)}{s\mu_F}$.*

Proof. By definition, $\tilde{F}_e(s) = \mathbb{E}\left[e^{-sX}\right]$, where X is a random variable with distribution function $F_e(x)$. We use integration by parts, to write

$$\tilde{F}_e(s) = \int_0^\infty e^{-sx}dF_e(x) = \frac{1}{s\mu_F} - \frac{1}{s\mu_F}\int_0^\infty e^{-sx}dF(x) = \frac{1}{s\mu}(1-\tilde{F}(s)).$$

□

Definition 1.3. A delayed renewal process with the initial arrival distribution $G = F_e$ is called the **equilibrium renewal process**.

Remark 1. Observe that F_e is the limiting distribution of the age and the excess time for the renewal process with common inter-renewal distribution F. Hence, if we start observing a renewal process at some arbitrarily large time t, then the observed renewal process is the equilibrium renewal process. This delayed renewal process exhibits stationary properties. That is, the limiting behaviors are exhibited for all times.

Theorem 1.4 (Renewal function). *The renewal function m_t^e for the equilibrium renewal process is linear for all times. That is, $m_t^e = \frac{t}{\mu_F}$.*

Proof. We know that the Laplace transform of renewal function $m_e(t)$ is given by

$$\tilde{m}^e(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)} = \frac{\tilde{F}_e(s)}{1-\tilde{F}(s)} = \frac{1}{s\mu_F}.$$

Further, we know that the Laplace transform of function t/μ is given by $\mathcal{L}_{t/\mu}(s) = \frac{1}{\mu}\int_0^\infty e^{-sx}dx = \frac{1}{s\mu}$. Since moment generating function is a one-to-one map, $m_t^e = \frac{t}{\mu_F}$ is the unique renewal function. □

Theorem 1.5 (Age and excess time). *The distribution of age A_t^e and excess time Y_t^e for the equilibrium renewal process are stationary. In particular, for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$, we have*

$$P\{A_t^e > x\} = \mathbb{1}_{\{t>x\}}\bar{F}_e(x), \qquad P\{Y_t^e > x\} = \bar{F}_e(x).$$

Proof. Recall that the age process A^e and the excess time process Y^e are delayed regenerative processes and $dm_t^e = \frac{1}{\mu_F}$. The solution of renewal equation for any equilibrium regenerative process is $f = K_1 + K_2 * m^e$. We can define function $t \mapsto f_t \triangleq P\{A_t^e > x\}$ and its kernel functions

$$t \mapsto K_1(t) \triangleq P\{A_t^e > x, X_1 > t\} = \mathbb{1}_{\{t>x\}}\bar{F}_e(t), \quad t \mapsto K_2(t) \triangleq P\left\{A_{S_1+t}^e > x, X_2 > t\right\} = \bar{F}(t)\mathbb{1}_{\{t>x\}}.$$

We can write the marginal distribution of equilibrium age process as

$$P\{A_t^e > x\} = \mathbb{1}_{\{t>x\}}(\bar{F}_e(t) + \frac{1}{\mu_F}\int_x^t \bar{F}(u)du) = \mathbb{1}_{\{t>x\}}(\bar{F}_e(t) + \bar{F}_e(x) - \bar{F}_e(t)) = \mathbb{1}_{\{t>x\}}\bar{F}_e(x).$$

Similarly, for $t \mapsto f_t \triangleq P\{Y^e_t > x\}$, we can write the kernel functions as $K_1(t) = \bar{F}_e(t+x)$ and $K_2(t) = \bar{F}(t+x)$. Thus, we can write the marginal distribution of equilibrium excess time process as

$$P\{Y^e_t > x\} = \bar{F}_e(t+x) + \frac{1}{\mu_F}\int_0^t \bar{F}(t+x-u)du = \bar{F}_e(t+x) + \frac{1}{\mu_F}\int_x^{t+x} \bar{F}(y)dy = \bar{F}_e(x).$$

□

Remark 2. When we start observing the counting process at time s, the observed renewal process is delayed renewal process with initial distribution Y^e at time s being identical to the distribution F_e. Hence, the number of renewals $N^e_{t+s} - N^e_s$ has the same distribution as N^e_t in duration $(0,t]$. That is, the distribution of counting process is shift invariant.

Theorem 1.6 (Stationary increments). *The counting process $N^e : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ for the equilibrium renewal process has stationary increments.*

Proof. We can write the event $\{N^e_{s+t} - N^e_s = n\} = \{S_{N^e_s+n} \leqslant t < S_{N^e_s+n+1}\}$ where $S_{N^e_s+n} = Y^e_s + \sum_{k=2}^n X_{N^e_s+k}$. Since Y^e_s is distributed identically to X_1, to show the result it suffices to show that $(X_{N^e_s+k} : k \geqslant 2)$ is *i.i.d.* , distributed identically to $(X_k : k \geqslant 2)$, and independent of Y^e_s. To this end, we consider the function $t \mapsto f_t \triangleq P\Big(\{Y^e_t > y\} \cap_{k=2}^n \Big\{X_{N^e_t+k} \leqslant x_k\Big\}\Big)$. Defining the kernel functions $t \mapsto k^1_t \triangleq P\Big(\{Y^e_t > y, S_1 > t\} \cap_{k=2}^n \Big\{X_{N^e_t+k} \leqslant x_k\Big\}\Big) = \bar{F}_e(t+y)\prod_{k=2}^n F(x_k)$, and $t \mapsto k^2_t \triangleq P\Big(\Big\{Y^e_{S_1+t} > y, t < X_2\Big\} \cap_{k=2}^n \Big\{X_{N^e_{S_1+t}+k} \leqslant x_k\Big\}\Big) = \bar{F}(t+y)\prod_{k=2}^n F(x_k)$. We can write the renewal function $f = k^1 + k^2 * m^e$ to obtain

$$P\Big(\{Y^e_t > y\} \cap_{k=2}^n \Big\{X_{N^e_t+k} \leqslant x_k\Big\}\Big) = \bar{F}_e(y)\prod_{k=2}^n F(x_k).$$

□

Example 1.7 (Poisson process). Consider the case, when inter-renewal time distribution F for a delay renewal process is exponential with rate λ. Here, one would expect the equilibrium distribution $F_e = F$, since Poisson process has stationary and independent increments. We observe that

$$F_e(x) = \frac{1}{\mu}\int_0^x \bar{F}(y)dy = \lambda\int_0^x e^{-\lambda y}dy = 1 - e^{-\lambda x} = F(x).$$

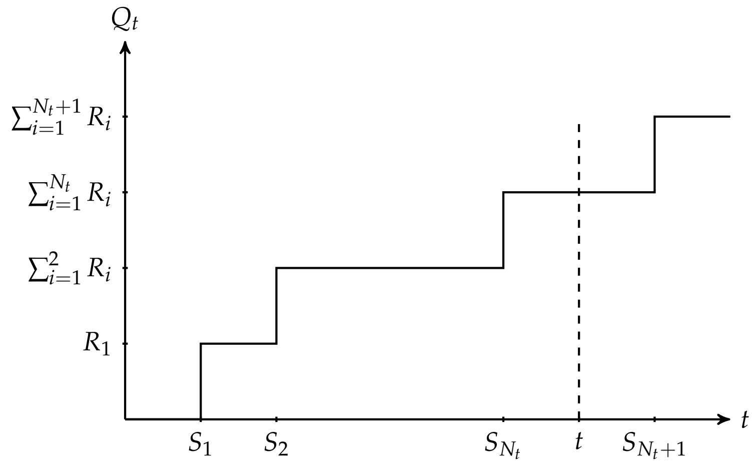
We see that F_e is also distributed exponentially with rate λ. Indeed, this is a Poisson process with rate λ.

2 Renewal reward process

Definition 2.1. Consider a counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, where the *i.i.d.* inter-renewal time sequence is denoted by $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having common distribution F. At the end of each renewal interval $n \in \mathbb{N}$, a random reward $R_n : \Omega \to \mathbb{R}$ is earned at time S_n, where the reward R_n is possibly dependent on the duration X_n. Let $(X, R) : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ be *i.i.d.* , then the **reward process** $Q : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ is defined as the accumulated reward earned by time t as $Q_t \triangleq \sum_{i=1}^{N_t} R_i$.

Example 2.2. Consider a renewal sequence $S : \Omega \to \mathbb{R}+^{\mathbb{N}}$ with *i.i.d.* inter-renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. Consider an *i.i.d.* renewal sequence $R : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ defined as $R_n \triangleq 1$ for all $n \in \mathbb{N}$. Then the reward process $Q : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is the same as the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with the renewal sequence S.

Example 2.3. Consider a renewal sequence $S : \Omega \to \mathbb{R}+^{\mathbb{N}}$ with *i.i.d.* inter-renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. Consider an *i.i.d.* renewal sequence $R : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ defined as $R_n \triangleq X_n$ for all $n \in \mathbb{N}$. Then the reward process $Q : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is the last renewal instant $Q_t = S_{N_t}$ for all times $t \in \mathbb{R}_+$.



Theorem 2.4 (renewal reward). *Consider a renewal reward process* $Q : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ *with associated* i.i.d. *renewal reward sequence* $(X, R) : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ *where the mean of absolute value of reward* $\mathbb{E}\,|R_1|$ *and mean of absolute value of renewal duration* $\mathbb{E}\,|X_1|$ *are finite. Then the empirical average of reward converges, almost surely and in mean, i.e.*

$$\lim_{t\to\infty} \frac{Q_t}{t} = \frac{\mathbb{E}R_1}{\mathbb{E}X_1} \text{ a.s. }, \qquad \lim_{t\to\infty} \frac{\mathbb{E}Q_t}{t} = \frac{\mathbb{E}R_1}{\mathbb{E}X_1}.$$

Proof. We can write the rate of accumulated reward as $\frac{Q_t}{t} = \left(\frac{Q_t}{N_t}\right)\left(\frac{N_t}{t}\right)$. From the strong law of large numbers we obtain that, $\lim_{t\to\infty} \frac{1}{N_t}\sum_{i=1}^{N_t} R_i = \mathbb{E}R_1$, and from the strong law for counting processes we have $\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}X_1}$.

Since $N_t + 1$ is a stopping time for the renewal reward sequence $((X_1, R_1), (X_2, R_2), \dots)$, it follows from Wald's lemma,

$$\mathbb{E}Q_t = \mathbb{E}\sum_{i=1}^{N_t} R_i = \mathbb{E}\sum_{i=1}^{N_t+1} R_i - \mathbb{E}R_{N_t+1} = (m_t + 1)\mathbb{E}R_1 - \mathbb{E}R_{N_t+1}.$$

Defining $t \mapsto g_t \triangleq \mathbb{E}R_{N_t+1}$, using elementary renewal theorem, it suffices to show that $\lim_{t\to\infty} \frac{g_t}{t} = 0$. Observe that R_{N_t+1} is a regenerative process with the regenerative sequence being the renewal instants S, since the nth segment is $\zeta_n \triangleq (X_n, R_n)$ and the sequence (X, R) is *i.i.d.* . Defining kernel function $t \mapsto K_t \triangleq \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1>t\}}]$, we can write the renewal function for g as

$$g_t = \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1>t\}}] + \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1\leqslant t\}}] = K_t + \int_0^t g_{t-u}dF(u).$$

Using the solution to renewal function, we can write $g = (1 + m) * K$ in terms of renewal function m and kernel function K. Using the conditional Jensen's inequality for convex function absolute, we observe that the kernel function $K : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded above as

$$K_t \triangleq \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1>t\}}] = \mathbb{E}[\mathbb{E}[R_1\mathbb{1}_{\{X_1>t\}} \mid \sigma(X_1)]] \leqslant \mathbb{E}[\mathbb{E}[|R_1|\,\mathbb{1}_{\{X_1>t\}} \mid \sigma(X_1)]]$$

From finiteness of $\mathbb{E}|R|$, it follows that $\lim_{t\to\infty} K_t = 0$, and we can choose T such that $|K_u| \leqslant \epsilon$ for all $u \geqslant T$. Hence, for all $t \geqslant T$, we have

$$\frac{|g_t|}{t} \leqslant \frac{|K_t|}{t} + \int_0^{t-T} \frac{|K_{t-u}|}{t} dm_u + \int_{t-T}^{t} \frac{|K_{t-u}|}{t} dm_u \leqslant \frac{\epsilon}{t} + \frac{\epsilon m_{t-T}}{t} + \mathbb{E}\,|R_1|\,\frac{(m_t - m_{t-T})}{t}.$$

Taking limits and applying elementary renewal and Blackwell's theorem, we get $\limsup_{t\to\infty} \frac{|g_t|}{t} \leqslant \frac{\epsilon}{\mathbb{E}X_1}$. The result follows since $\epsilon > 0$ was arbitrary. □

Corollary 2.5. *Renewal reward theorem applies to a reward process* Q *that accrues positive reward continuously over a renewal duration. The total reward in a renewal duration* X_n *remains* R_n *as before, with the sequence*$(X, R) : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ *being* i.i.d. .

Proof. Let the process $t \mapsto Q_t$ denote the accumulated reward until time t, when the reward accrual is continuous in time. Defining $R_n \triangleq Q_{S_n} - Q_{S_{n-1}} > 0$, it follows that

$$\frac{\sum_{n=1}^{N_t} R_n}{t} \leqslant \frac{Q_t}{t} \leqslant \frac{\sum_{n=1}^{N_t+1} R_n}{t}.$$

First result follows from application of strong law of large numbers. For the second result, we have

$$\frac{(m_t+1)\mathbb{E}R_1 - \mathbb{E}R_{N_t+1}}{t} \leqslant \frac{\mathbb{E}Q_t}{t} \leqslant \frac{(m_t+1)\mathbb{E}R_1}{t}.$$

The second result follows from the fact that $\lim_{t\to\infty} \frac{\mathbb{E}R_{N_t+1}}{t} = 0$. □