Lecture-16: Invariant Distribution

1 Invariant distribution

Definition 1.1. For a time-homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with transition matrix P, a distribution $\pi \in \mathcal{M}(\mathcal{X})$ is called **invariant** if it is a left eigenvector of the probability transition matrix P with eigenvalue unity, or

$$\pi = \pi P.$$

Remark 1. Recall that $\nu(n) \in \mathcal{M}(\mathcal{X})$ where $\nu_x(n) = P\{X_n = x\}$ for all $x \in \mathcal{X}$, denotes the probability distribution of the Markov chain X being in one of the states at step $n \in \mathbb{N}$. Then, if $\nu(0) = \pi$, then $\nu(n) = \nu(0)P^n = \pi$ for all time-steps $n \in \mathbb{N}$.

Definition 1.2. For a time-homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with transition matrix P, the **stationary distribution** is defined as $\nu(\infty) \triangleq \lim_{n\to\infty} \nu(n)$.

Remark 2. For a Markov chain with initial distribution being invariant, the stationary distribution is invariant distribution.

Example 1.3 (Simple random walk on a directed graph). Let $G = (\mathcal{X}, E)$ be a finite directed graph. We define a simple random walk on this graph as a Markov chain with state space $\mathcal{X}$ and transition matrix $P : \mathcal{X} \times \mathcal{X} \to [0,1]$ where $p_{xy} \triangleq \frac{1}{\deg_{\text{in}}(x)} \mathbb{1}_{\{(x,y)\in E\}}$. We observe that vector $(\deg_{\text{in}}(x) : x \in \mathcal{X})$ is a left eigenvector of the transition matrix P with unit eigenvalue. Indeed we can very that

$$\sum_{x\in\mathcal{X}} \deg_{\text{in}}(x) p_{xy} = \sum_{x\in\mathcal{X}} \mathbb{1}_{\{(x,y)\in E\}} = \deg_{\text{in}}(y).$$

Since $\sum_{x\in\mathcal{X}} \deg_{\text{in}}(x) = 2|E|$, it follows that $\pi : \mathcal{X} \to [0,1]$ defined by $\pi_x \triangleq \frac{\deg_{\text{in}}(x)}{2|E|}$ for each $x \in \mathcal{X}$, is the equilibrium distribution of this simple random walk.

1.1 Existence of an invariant distribution

Proposition 1.4. *Consider an irreducible and aperiodic homogeneous DTMC $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with transition matrix P and starting from initial state $X_0 = x$. Let the positive vector $\tilde{\pi}_x : \mathcal{X} \to \mathbb{R}_+$ defined as $\tilde{\pi}_x(y) \triangleq \mathbb{E}_x N_y(\tau_x^+)$ for all $y \in \mathcal{X}$. Then the following statements hold true.*
(a) If x is recurrent, then $\tilde{\pi}_x$ is a left eigenvector of P with eigenvalue unity. That is, $\tilde{\pi}_x = \tilde{\pi}_x P$.
(b) If x is positive recurrent, then $\pi \triangleq \frac{\tilde{\pi}_x}{\mathbb{E}_x \tau_x^+}$ is an invariant distribution of P.

Proof. (a) We first show that $\tilde{\pi}_x$ is a left eigenvector for the transition probability matrix P for time homogeneous DTMC X. Recall that $N_y(\tau_x^+) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{n\leqslant\tau_x^+\}} \mathbb{1}_{\{X_n=y\}}$ and $p_{wz} = P(\{X_{n+1} = z\} \mid \{X_n = w\})$. Using the monotone convergence theorem, we can write

$$\sum_{w\in\mathcal{X}} \tilde{\pi}_x(w) p_{wz} = \sum_{w\in\mathcal{X}} \sum_{n\in\mathbb{N}} P_x\left\{\tau_x^+ \geqslant n, X_n = w\right\} P(\{X_{n+1} = z\} \mid \{X_n = w\}).$$

We first focus on the term $w = x$. We see that $\{X_n = x, \tau_x^+ \geqslant n\} = \{\tau_x^+ = n\}$ and hence for a recurrent state x, we have

$$\tilde{\pi}_x(x) p_{xz} = p_{xz} \sum_{n\in\mathbb{N}} P_x\left\{\tau_x^+ = n\right\} = p_{xz} P_x\left\{\tau_x^+ < \infty\right\} = p_{xz}.$$

We next focus on the terms $w \neq x$. We observe that $\{X_n = w, \tau_x^+ \geqslant n\} = \{X_n = w, \tau_x^+ \geqslant n+1\} \in \mathcal{F}_n$. From the Markov property for X, we have

$$p_{wz} = P(\{X_{n+1} = z\} \mid \{X_n = w\}) = P(\{X_{n+1} = z\} \mid \{X_n = w, \tau_x^+ \geqslant n+1, X_0 = x\}).$$

Therefore, from the definition of conditional probability, we obtain $p_{wz} P_x \{X_n = w, \tau_x^+ \geqslant n+1, X_0 = x\} = P_x \{X_{n+1} = z, X_n = w, \tau_x^+ \geqslant n+1\}$, and hence

$$\sum_{w \neq x} \tilde{\pi}_x(w) p_{wz} = \sum_{n \in \mathbb{N}} \sum_{w \neq x} P_x \{X_n = w, X_{n+1} = z, \tau_x^+ \geqslant n+1\} = \sum_{n \in \mathbb{N}} P_x \{X_{n+1} = z, \tau_x^+ \geqslant n+1\}$$
$$= \tilde{\pi}_x(z) - P_x \{X_1 = z, \tau_x^+ \geqslant 1\} = \tilde{\pi}_x(z) - p_{xz}.$$

The result follows from summing both the cases.

(b) For a positive recurrent state x, it suffices to show that π is a distribution on state space $\mathcal{X}$. Recall that $\tilde{\pi}_x(y) = \mathbb{E}_x N_y(\tau_x^+)$ and $\sum_{y \in \mathcal{X}} N_y(\tau_x^+) = \tau_x^+$, it follows from the monotone convergence theorem that

$$\sum_{y \in \mathcal{X}} \tilde{\pi}_x(y) = \mathbb{E}_x \sum_{y \in \mathcal{X}} N_y(\tau_x^+) = \mathbb{E}_x \tau_x^+ < \infty.$$

Thus $0 \leqslant \tilde{\pi}_x(y) \leqslant \mathbb{E}_x \tau_x^+$ for all states $y \in \mathcal{X}$. Further, $N_x(\tau_x^+) = 1$ and hence $\tilde{\pi}_x(x) = 1$. It follows that $\frac{\tilde{\pi}_x}{\mathbb{E}_x \tau_x^+}$ is a distribution on the state space $\mathcal{X}$.

□

1.2 Uniqueness of the invariant distribution

Definition 1.5. Consider a time homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with transition probability matrix $P \in [0,1]^{\mathcal{X} \times \mathcal{X}}$. A function $h : \mathcal{X} \to \mathbb{R}$ is called **harmonic at** x if $h(x) = \mathbb{E}_x h(X_1) = (Ph)_x = \sum_{y \in \mathcal{X}} p_{xy} h(y)$. A function is **harmonic on a subset** $D \subset \mathcal{X}$ if it is harmonic at every state $x \in D$.

Lemma 1.6. *For a finite state irreducible Markov chain, a function f harmonic on all states in $\mathcal{X}$ is a constant.*

Proof. Suppose h is not a constant, then there exists a state $x \in \mathcal{X}$, such that $h(x) \geqslant h(y)$ for all states $y \in \mathcal{X}$. Since the Markov chain is irreducible, there exists a state $z \in \mathcal{X}$ such that $p_{xz} > 0$. Let's assume $h(z) < h(x)$, then

$$h(x) = p_{xz} h(z) + \sum_{y \neq z} p_{xy} h(y) < h(x).$$

This implies that $h(x) = h(z)$ for all states z such that $p_{xz} > 0$. By induction, this implies that $h(x) = h(y)$ for any state y reachable from state x. Since all states are reachable from state x by irreducibility, this implies h is a constant on the state space $\mathcal{X}$. □

Corollary 1.7. *For any irreducible and aperiodic finite state Markov chain, there exists a unique invariant distribution π.*

Proof. For an aperiodic and irreducible DTMC $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with finite state space $\mathcal{X}$, we have $\mathbb{E}_x \tau_y^+ < \infty$ for all states $x, y \in \mathcal{X}$. In particular, X is positive recurrent and hence there exists a positive invariant distribution π. Further, from previous Lemma we have that the dimension of null-space of $(P - I)$ is unity. Hence, the rank of $P - I$ is $|\mathcal{X}| - 1$. Therefore, all vectors satisfying $\nu = \nu P$ are scalar multiples of π. □

1.3 Stationary distribution for irreducible and aperiodic finite DTMC

Theorem 1.8. *For a finite state irreducible and aperiodic Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, its invariant distribution is same as its stationary distribution.*

Proof. For a finite state irreducible and aperiodic DTMC $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, we have $\mathbb{E}_x \tau_y^+ < \infty$ and hence $P_x \left\{ \tau_y^+ < \infty \right\} = 1$ for all $x, y \in \mathcal{X}$. That is, the return times are finite almost surely, and hence we can apply strong Markov property at these stopping times to obtain that DTMC X is a regenerative process with delayed renewal sequence $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$, where $\tau_y^+(0) \triangleq 0$, and $\tau_y^+(n) \triangleq \inf \left\{ m > \tau_y^+(n-1) : X_m = y \right\}$. We can create an on-off alternating renewal function on this DTMC X, which is ON when in state y. Then, from the limiting ON probability of alternating renewal function, we know that

$$\pi(y) \triangleq \lim_{n \to \infty} P_x \{X_n = y\} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{X_k = y\}} = \frac{1}{\mathbb{E}_y \tau_y^+}.$$

We observe that $\pi(y) = \frac{\tilde{\pi}_y(y)}{\mathbb{E}_y \tau_y^+}$ for each state $y \in \mathcal{X}$. From the uniqueness of invariant distribution, it follows that π is the unique invariant distribution of the DTMC X. We observe that $\pi(x)$ is the long-term average of the amount of time spent in state x and from renewal reward theorem $\pi(x) = \frac{1}{\mathbb{E}_x \tau_x^+}$. $\square$