Lecture-23: Reversed Processes

1 Reversed Processes

Definition 1.1. Let $X : \Omega \to \mathcal{X}^T$ be a stochastic process with index set T being an additive ordered group such as $\mathbb{R}$ or $\mathbb{Z}$. Then, $\hat{X}^\tau : \Omega \to \mathcal{X}^T$ defined as $\hat{X}^\tau_t \triangleq X_{\tau-t}$ for all $t \in T$ is the **reversed process** for some $\tau \in T$.

Remark 1. Note that a reversed process, doesn't have to have the identical distribution to the original process. For a reversible process X, the reversed process would have identical distribution.

Lemma 1.2. *If $X : \Omega \to \mathcal{X}^T$ is a Markov process, then the reversed process $\hat{X}^\tau$ is also Markov for any $\tau \in T$.*

Proof. Let $\mathcal{F}_\bullet$ be the natural filtration of the process X. From the Markov property of process X, for any event $F \in \sigma(X_u : u > t), H \in \sigma(X_s : s < t)$, states $x, y \in \mathcal{X}$, and times $u, s > 0$, we have

$$P(F \mid \{X_t = y\} \cap H) = P(F \mid \{X_t = y\}).$$

Markov property of the reversed process follows from the observation, that

$$P(H \mid \{X_t = y\} \cap F) = \frac{P(H \cap \{X_t = y\})P(F \mid H \cap \{X_t = y\})}{P\{X_t = y\} P(F \mid \{X_t = y\})} = P(H \mid \{X_t = y\}).$$

□

Remark 2. Even if the forward process X is time-homogeneous, the reversed process need not be time-homogeneous. For a non-stationary time-homogeneous Markov process, the reversed process is Markov but not necessarily time-homogeneous.

Theorem 1.3. *If $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ is an irreducible, positive recurrent, stationary, and homogeneous Markov process with transition kernel $P : \mathbb{R} \to [0,1]^{\mathcal{X}\times\mathcal{X}}$ and invariant distribution $\pi \in \mathcal{M}(\mathcal{X})$, then the reversed Markov process $\hat{X}^\tau : \Omega \to \mathcal{X}^{\mathbb{R}}$ is also irreducible, positive recurrent, stationary, and homogeneous with the same invariant distribution π and transition kernel $\hat{P} : \mathbb{R} \to [0,1]^{\mathcal{X}\times\mathcal{X}}$ defined for all $t \in T$ and states $x, y \in \mathcal{X}$, as*

$$\hat{P}_{xy}(t) \triangleq \frac{\pi_y}{\pi_x} P_{yx}(t).$$

Further, for any finite sequence of states $x \in \mathcal{X}^n$, finite sequence of times $t \in T^n$ such that $t_1 < t_2 < \cdots < t_n$, and any shift $\tau \in \mathbb{R}$, we have

$$P_\pi \cap_{i=1}^n \{X_{t_i} = x_i\} = \hat{P}_\pi \cap_{i=1}^n \left\{\hat{X}^\tau_{t_i} = x_i\right\}.$$

Proof. We observe that $\hat{X}^\tau_{t_i} = X_{\tau - t_i}$ for all $i \in [n]$.

Step 1: We can verify that $\hat{P}$ is a probability transition kernel.

- $\hat{P}_{xy} \geqslant 0$ for all $t \in T$.
- $\sum_{y\in\mathcal{X}} \hat{P}_{xy}(t) = \frac{1}{\pi_x}\sum_{y\in\mathcal{X}} \pi_y P_{yx}(t) = 1.$

Step 2: π is an invariant distribution for $\hat{P}$, since for all states $x, y \in \mathcal{X}$

$$\sum_{x\in\mathcal{X}} \pi_x \hat{P}_{xy}(t) = \pi_y \sum_{x\in\mathcal{X}} P_{yx}(t) = \pi_y.$$

Step 3: We next wish to show that $\hat{P}$ defined in the Theorem, is the probability transition kernel for the reversed process. Since the forward process is stationary and time-homogeneous, we can write the probability transition kernel for the reversed process as

$$P(\{\hat{X}^\tau_{t+s} = y\} \mid \{\hat{X}^\tau_s = x\}) = \frac{P_\pi\{X_{\tau-t-s} = y, X_{\tau-s} = x\}}{P_\pi\{X_{\tau-s} = x\}} = \frac{\pi_y P_{yx}(t)}{\pi_x}.$$

This implies that the reversed process is time-homogeneous and has the desired probability transition kernel. Further, π is the invariant distribution for the reversed process and is the marginal distribution for the reversed process at any time t, and hence the reversed process is also stationary.

Step 4: We now need to check if $\hat{P}$ is irreducible. For an irreducible and positive recurrent Markov process with invariant distribution π, we have $\pi_x > 0$ for each state $x \in \mathcal{X}$. Since the forward process is irreducible, there exists a time $t \geqslant 0$ such that $P_{yx}(t) > 0$ for states $x, y \in \mathcal{X}$, and hence $\hat{P}_{xy}(t) > 0$ implying irreducibility of the reversed process.

Step 5: Now, we want to show that $P_\pi \cap_{i=1}^n \{X_{t_i} = x_i\} = \hat{P}_\pi \cap_{i=1}^n \left\{\hat{X}^\tau_{t_i} = x_i\right\}$. From the Markov property of the underlying processes and definition of $\hat{P}$, we can write

$$P_\pi\Big(\cap_{i=1}^n \{X_{t_i} = x_i\}\Big) = \pi_{x_1} \prod_{i=1}^{n-1} P_{x_i x_{i+1}}(t_{i+1} - t_i) = \pi_{x_n} \prod_{i=1}^{n-1} \hat{P}_{x_{i+1} x_i}((\tau - t_i) - (\tau - t_{i+1})) = \hat{P}_\pi\Big(\cap_{i=1}^n \left\{\hat{X}^\tau_{t_i} = x_i\right\}\Big).$$

This follows from the fact that $\pi_{x_1} P_{x_1 x_2}(t_2 - t_1) = \pi_{x_2} \hat{P}_{x_2 x_1}(t_2 - t_1)$, and hence we have

$$\pi_{x_1} \prod_{i=1}^{n-1} P_{x_i x_{i+1}}(t_{i+1} - t_i) = \pi_{x_n} \prod_{i=1}^{n-1} \hat{P}_{x_{i+1} x_i}(t_{i+1} - t_i).$$

For any finite $n \in \mathbb{N}$, we see that the joint distributions of $(X_{t_1}, \dots, X_{t_n})$ and $(X_{s+t_1}, \dots, X_{s+t_n})$ are identical for all $s \in T$, from the stationarity of the process X. It follows that $\hat{X}^\tau$ is also stationary, since $(\hat{X}^\tau_{t_n}, \dots, \hat{X}^\tau_{t_1})$ and $(\hat{X}^\tau_{s+t_n}, \dots, \hat{X}^\tau_{s+t_1})$ have the identical distribution.

□

1.1 Reversed Markov Chain

Corollary 1.4. *If $X : \Omega \to \mathcal{X}^{\mathbb{Z}}$ is an irreducible, stationary, homogeneous Markov chain with transition matrix P and invariant distribution π, then the reversed chain $\hat{X}^\tau : \Omega \to \mathcal{X}^{\mathbb{Z}}$ is an irreducible stationary, time homogeneous Markov chain with the same invariant distribution π, and transition matrix $\hat{P}$ defined as $\hat{P}_{xy} \triangleq \frac{\pi_y}{\pi_x} P_{yx}$, for all $x, y \in \mathcal{X}$.*

Corollary 1.5. *Consider an irreducible Markov chain X with transition matrix $P : \mathcal{X} \times \mathcal{X} \to [0,1]$. If one can find a non-negative vector $\alpha \in \mathcal{M}(\mathcal{X})$ and other transition matrix $P^* : \mathcal{X} \times \mathcal{X} \to [0,1]$ that satisfies the detailed balance equation for all $x, y \in \mathcal{X}$*

$$\alpha_x P_{xy} = \alpha_y P^*_{yx}, \tag{1}$$

then P^* is the transition matrix for the reversed chain and α is the invariant distribution for both chains.

Proof. Summing both sides of the detailed balance equation (1) over x, we obtain that α is the invariant distribution of the forward chain. Since $P^*_{yx} = \frac{\alpha_x P_{xy}}{\alpha_y}$, it follows that $P^* : \mathcal{X} \times \mathcal{X} \to [0,1]$ is the transition matrix of the the reversed chain and α is the invariant distribution of the reversed chain. □

1.2 Reversed Markov Process

Corollary 1.6. *If $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ is an irreducible, stationary, homogeneous Markov process with generator matrix Q and invariant distribution π, then the reversed process $\hat{X}^\tau : \Omega \to \mathcal{X}^{\mathbb{R}}$ is also an irreducible, stationary, homogeneous Markov process with same invariant distribution π and generator matrix $\hat{Q}$ defined as $\hat{Q}_{xy} \triangleq \frac{\pi_y}{\pi_x} Q_{yx}$, for all $x, y \in \mathcal{X}$.*

Corollary 1.7. *Let $Q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ denote the rate matrix for an irreducible Markov process. If we can find $Q^* : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and a distribution $\pi \in \mathcal{M}(\mathcal{X})$ such that for $y \neq x \in \mathcal{X}$, we have*

$$\pi_x Q_{xy} = \pi_y Q^*_{yx}, \qquad \text{and} \qquad \sum_{y \neq x} Q_{xy} = \sum_{y \neq x} Q^*_{xy},$$

then Q^* is the rate matrix for the reversed Markov process and π is the invariant distribution for both processes.

2 Applications of Reversed Processes

2.1 Truncated Markov Processes

Definition 2.1. For a Markov process $X : \Omega \to \mathcal{X}^{\mathbb{R}}$, and a subset $A \subseteq \mathcal{X}$ the boundary of A is defined as

$$\partial A \triangleq \left\{ y \notin A : Q_{xy} > 0, \text{ for some } x \in A \right\}.$$

Example 2.2. Consider a birth-death process. Let $A = \{3,4\}$. Then, $\partial A = \{2,5\}$

Definition 2.3. Consider a transition rate matrix $Q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ on the countable state space $\mathcal{X}$. Given a nonempty subset $A \subseteq \mathcal{X}$, the truncation of Q to A is the transition rate matrix $Q^A : A \times A \to \mathbb{R}$, where for all $x, y \in A$

$$Q^A_{xy} \triangleq \begin{cases} Q_{xy}, & y \neq x, \\ -\sum_{z \in A \setminus \{x\}} Q_{xz}, & y = x. \end{cases}$$

Proposition 2.4. *Suppose $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ is an irreducible, time-reversible CTMC on the countable state space $\mathcal{X}$, with generator $Q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and invariant distribution $\pi \in \mathcal{M}(\mathcal{X})$. Suppose the truncated Markov process X^A to a set of states $A \subseteq \mathcal{X}$ with generator matrix Q^A is irreducible. Then, $X^A : \Omega \to A^{\mathbb{R}}$ at stationarity is time-reversible, with invariant distribution $\pi^A \in \mathcal{M}(A)$ defined as*

$$\pi^A_y \triangleq \frac{\pi_y}{\sum_{x \in A} \pi_x}, \quad y \in A.$$

Proof. It is clear that π^A is a distribution on state space A. We must show the reversibility with this distribution π^A. That is, we must show for all states $x, y \in A$

$$\pi^A_x Q_{xy} = \pi^A_y Q_{yx}.$$

However, this is true since the original chain is time reversible. □

Example 2.5 (Limiting waiting room: M/M/1/K). Consider a variant of the $M/M/1$ queueing system with load $\rho \triangleq \frac{\lambda}{\mu}$ that has a finite buffer capacity of at most K customers. Thus, customers that arrive when there are already K customers present are *rejected*. It follows that the CTMC for this system is simply the $M/M/1$ CTMC truncated to the state space $\{0, 1, \dots, K\}$, and so it must be time-reversible with invariant distribution

$$\pi_i = \frac{\rho^i}{\sum_{j=0}^{k} \rho^j}, \quad 0 \leqslant i \leqslant k.$$

Example 2.6 (Two queues with joint waiting room). Consider two independent $M/M/1$ queues with arrival and service rates λ_i and μ_i respectively for $i \in [2]$. Then, the joint distribution of two queues is

$$\pi(n_1, n_2) = (1 - \rho_1)\rho_1^{n_1}(1 - \rho_2)\rho_2^{n_2}, \quad n_1, n_2 \in \mathbb{Z}_+.$$

Suppose both the queues are sharing a common waiting room, where if arriving customer finds R waiting customer then it leaves. Defining $A \triangleq \left\{ n \in \mathbb{Z}^2_+ : n_1 + n_2 \leqslant R \right\}$, we observe that the joint Markov porcess is restricted to the set of states A, and the invariant distribution for the truncated Markov process is

$$\pi(n_1, n_2) = \frac{\rho_1^{n_1} \rho_2^{n_2}}{\sum_{(m_1, m_2) \in A} \rho_1^{m_1} \rho_2^{m_2}}, \quad (n_1, n_2) \in A.$$