# Lecture-26: Martingale Convergence Theorem

## 1 Martingale Convergence Theorem

Before we state and prove martingale convergence theorem, we state some results which will be used in the proof of the theorem.

**Lemma 1.1.** *If $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ is a submartingale and $\tau : \Omega \to \mathbb{N}$ is a stopping time with respect to a filtration $\mathcal{F}_\bullet$, such that there exists some $N \in \mathbb{N}$ such that $P\{\tau \leqslant N\} = 1$. Then*

$$\mathbb{E}X_1 \leqslant \mathbb{E}X_\tau \leqslant \mathbb{E}X_N.$$

*Proof.* Recall that for any random time $\tau$, the stopped process $X^\tau$ is submartingale Heence $\mathbb{E}X_\tau \geqslant \mathbb{E}X_1$. Since $\tau$ is a stopping time, we see that for the event $\{\tau = k\}$ for any $k \leqslant N$

$$\mathbb{E}[X_N \mathbb{1}_{\{\tau=k\}} | \mathcal{F}_k] \geqslant X_k \mathbb{1}_{\{\tau=k\}} = X_\tau \mathbb{1}_{\{\tau=k\}}.$$

Result follows by taking expectation on both sides and summing over $k$. That is,

$$\mathbb{E}X_N = \mathbb{E}\sum_{k=1}^{N} X_N \mathbb{1}_{\{\tau=k\}} \geqslant \mathbb{E}\sum_{k=1}^{N} X_\tau \mathbb{1}_{\{\tau=k\}} = \mathbb{E}X_\tau.$$

□

**Definition 1.2.** Consider a discrete random process $X : \Omega \to \mathbb{R}^{\mathbb{Z}_+}$ adapted to the filtration $\mathcal{F}_\bullet = (\mathcal{F}_n \subseteq \mathcal{F} : n \in \mathbb{Z}_+)$. Let $N_0 \triangleq 0$. For the two thresholds $a < b$, we define the stopping times corresponding to $k$th downcrossing and upcrossing times as

$$N_{2k-1} \triangleq \inf\{m > N_{2k-2} : X_m \leqslant a\}, \qquad N_{2k} \triangleq \inf\{m > N_{2k-1} : X_m \geqslant b\}.$$

We next define the indicator to the event that the process is in $k$th upcrossing transition from $a$ to $b$ at time $m$,

$$H_m \triangleq \sum_{k\in\mathbb{N}} \mathbb{1}_{\{N_{2k-1} < m \leqslant N_{2k}\}}.$$

The number of upcrossings completed in time $n$ is defined by

$$U_n \triangleq \sup\{k \in \mathbb{N} : N_{2k} \leqslant n\} = \sum_{k\in\mathbb{N}} \mathbb{1}_{\{N_{2k} \leqslant n\}}.$$

*Remark* 1. For each $k \in \mathbb{N}$, $N_{2k}, N_{2k-1}$ are integer stopping times, and hence we have

$$\{N_{2k-1} < m \leqslant N_{2k}\} = \{N_{2k-1} \leqslant m-1\} \cap \{N_{2k} \leqslant m-1\}^c \in \mathcal{F}_{m-1}.$$

It follows that $\sigma(H_m) \subseteq \mathcal{F}_{m-1}$. Hence, the event that the process $X$ is in an upcrossing transition at time $m$ is predictable. Since $N_0 = 0$, it follows that $N_1 \geqslant 1$ and $H_1 = 0$.

**Lemma 1.3 (Upcrossing inequality).** *Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be a submartingale with respect to a filtration $\mathcal{F}_\bullet$. Then, we have*

$$(b-a)\mathbb{E}U_n \leqslant \mathbb{E}(X_n - a)_+$$

*Proof.* Define a random sequence $Y : \Omega \to \mathbb{R}^{\mathbb{N}}$ such that $Y_n \triangleq a + (X_n - a)^+ = X_n \vee a$ for each $n \in \mathbb{N}$. Since $x \mapsto f(x) = x \wedge a$ is a convex function and $X$ is a submartingale w.r.t. $\mathcal{F}_\bullet$, $Y$ is also a submartingale with respect to $\mathcal{F}_\bullet$. Since each upcrossing has a gain lower bounded by $b - a$, we get

$$(H \cdot Y)_n = \sum_{m=1}^{n} \sum_{k\in\mathbb{N}} \mathbb{1}_{\{N_{2k-1} < m \leqslant N_{2k}\}}(Y_m - Y_{m-1}) = \sum_{k=1}^{U_n}(Y_{N_{2k}} - Y_{N_{2k-1}}) \geqslant (b-a)U_n.$$

Let $K_m \triangleq 1 - H_m$ for each $m \in \mathbb{N}$. Since $H$ is predictable, then so is $K$ with respect to $\mathcal{F}_\bullet$, and

$$Y_n - Y_0 = \sum_{i=1}^{n}(Y_i - Y_{i-1}) = \sum_{i=1}^{n}(H_i + K_i)(Y_i - Y_{i-1}) = (H \cdot Y)_n + (K \cdot Y)_n.$$

Since $H : \Omega \to \{0,1\}^{\mathbb{N}}$ is a non-negative and bounded sequence, so is $K : \Omega \to \{0,1\}^{\mathbb{N}}$. Further, since $Y$ is a submartingale, so is $((K \cdot Y)_n : n \in \mathbb{Z}_+)$. Therefore, we can write

$$\mathbb{E}[(K \cdot Y)_n] \geqslant \mathbb{E}[(K \cdot Y)_1] = \mathbb{E}[K_1(Y_1 - Y_0)] = \mathbb{E}[Y_1 - Y_0] \geqslant -\mathbb{E}(X_0 - a)^+.$$

Therefore, it follows that

$$\mathbb{E}(Y_n - Y_0) = \mathbb{E}(H \cdot Y)_n + \mathbb{E}(K \cdot Y)_n \geqslant \mathbb{E}(H \cdot Y)_n - \mathbb{E}(X_0 - a)^+ \geqslant (b-a)\mathbb{E}U_n - \mathbb{E}(X_0 - a)^+.$$

The result follows from the fact that $\mathbb{E}Y_n - \mathbb{E}Y_0 = \mathbb{E}(X_n - a)^+ - \mathbb{E}(X_0 - a)^+$. □

**Theorem 1.4 (Martingale convergence theorem).** *If $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ is a submartingale with respect to filtration $\mathcal{F}_\bullet$ such that $\sup_{n\in\mathbb{N}} \mathbb{E}X_n^+ < \infty$, then $\lim_{n\in\mathbb{N}} X_n = X_\infty$ a.s with $\mathbb{E}|X_\infty| < \infty$, i.e. $X$ converges almost surely in both value and mean.*

*Proof.* Since $(X - a)^+ \leqslant X^+ + |a|$, it follows from upcrossing inequality that

$$\mathbb{E}U_n \leqslant \frac{\mathbb{E}X_n^+ + |a|}{b-a}.$$

The number of upcrossings $U_n$ increases with $n$, however the mean $\mathbb{E}U_n$ is uniformly bounded above for each $n \in \mathbb{N}$. Hence, $\lim_{n\in\mathbb{N}} \mathbb{E}U_n$ exists and is finite.

Let $U \triangleq \lim_{n\in\mathbb{N}} U_n$ and since $\mathbb{E}U \leqslant \sup_n \frac{\mathbb{E}X_n^+ + |a|}{b-a} < \infty$, we have $U < \infty$ almost surely. This conclusion implies

$$P_{a,b\in\mathbb{Q}} \cup \left\{\liminf_{n\in\mathbb{N}} X_n < a < b < \limsup_{n\in\mathbb{N}} X_n\right\} = 0.$$

From the above probability, we have almost sure equality $\limsup_{n\in\mathbb{N}} X_n = \liminf_{n\in\mathbb{N}} X_n$. That is, the $\lim_{n\in\mathbb{N}} X_n$ exists almost surely.

Fatou's lemma guarantees

$$\mathbb{E}X_\infty^+ \leqslant \liminf_{n\in\mathbb{N}} \mathbb{E}X_n^+ < \infty,$$

which implies $X_\infty < \infty$ almost surely. From the submartingale property of $X_n$, it follows that

$$\mathbb{E}X_n^- = \mathbb{E}X_n^+ - \mathbb{E}X_n \leqslant \mathbb{E}X_n^+ - \mathbb{E}X_0.$$

From Fatou's lemma, we get

$$\mathbb{E}X_\infty^- \leqslant \liminf_{n\in\mathbb{N}} \mathbb{E}X_n^- \leqslant \sup_{n\in\mathbb{N}} \mathbb{E}X_n^+ - \mathbb{E}X_0 < \infty.$$

This implies $X_\infty > -\infty$ almost surely, completing the proof. □

**Example 1.5 (Polya's Urn Scheme).** Consider a discrete time stochastic process $((B_n, W_n) : n \in \mathbb{N})$, where $B_n, W_n$ respectively denote the number of black and white balls in an urn after $n \in \mathbb{N}$ draws. At each draw $n$, balls are uniformly sampled from this urn. After each draw, one additional ball of the same color to the drawn ball, is returned to the urn. We are interested in characterizing evolution of this urn, given initial urn content $(B_0, W_0)$. Let $\xi_i$ be a random variable indicating the outcome of the $i$th draw being a black ball. For example, if the first drawn ball is a black, then $\xi_1 = 1$ and $(B_1, W_1) = (B_0 + 1, W_0)$. In general,

$$B_n = B_0 + \sum_{i=1}^{n} \xi_i = B_{n-1} + \xi_n, \qquad W_n = W_0 + \sum_{i=1}^{n}(1 - \xi_i) = W_{n-1} + 1 - \xi_n.$$

It is clear that $B_n + W_n = B_0 + W_0 + n$. Let $\mathcal{F}_n = \sigma(B_0, W_0, \xi_1, \dots, \xi_n)$ be the $\sigma$-field generated by the first $n$ indicators to black draws. We are interested in limiting ratio of black balls. We represent the proportion of black balls after $n$ draws by

$$X_n = \frac{B_n}{B_n + W_n} = \frac{B_n}{B_0 + W_0 + n}.$$

It is clear that $\mathbb{E}[\xi_{n+1}|\mathcal{F}_n] = X_n$. Using this fact, we observe that $X : \Omega \to [0,1]^{\mathbb{N}}$ is a martingale adapted to filtration $\mathcal{F}_\bullet = (\mathcal{F}_n : n \in \mathbb{N})$, since

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \frac{1}{B_0 + W_0 + n + 1}\mathbb{E}[B_{n+1}|\mathcal{F}_n] = \frac{B_n + X_n}{\frac{B_n}{X_n} + 1} = X_n.$$

For each $n \in \mathbb{N}$, we have $\mathbb{E}X_n^+ = \mathbb{E}X_n \leqslant 1$. From Martingale convergence theorem, it follows that $\lim_{n\in\mathbb{N}} X_n$ exists almost surely. Further, from the Martingale property, we have for all $n \in \mathbb{N}$

$$\mathbb{E}X_n = X_0 = \frac{B_0}{B_0 + W_0}.$$

It follows that $\lim_{n\in\mathbb{N}} X_n = X_0$ almost surely.