Lecture-29: Random Walks

1 Introduction

Definition 1.1. Let $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ be an *i.i.d.* step-size sequence, where $\mathcal{X} \subseteq \mathbb{R}$ and $\mathbb{E}|X_1| < \infty$. We define $S_0 \triangleq 0$ and the location of a particle after n steps as $S_n \triangleq \sum_{i=1}^{n} X_i$. Then the sequence $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ is called a *random walk process*. If the step-size alphabet $\mathcal{X} = \{-1, 1\}$, then the random walk is called **simple**.

Remark 1. Random walks are generalizations of renewal processes. If X was a sequence of non-negative random variables indicating inter-renewal times, then S_n is the instant of the nth renewal event.

2 Duality in random walks

Lemma 2.1 (Duality principle). *For any finite $n \in \mathbb{N}$, the joint distributions of finite sequence $(X_1, X_2, \cdots, X_n)$ and the reversed sequence $(X_n, X_{n-1}, \cdots, X_1)$ are identical, for any* i.i.d. *step-size sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$.*

Proof. Since $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ is a sequence of *i.i.d.* random variables, it is exchangeable. The reversed sequence is $(X_{\sigma(1)}, \dots, X_{\sigma(n)})$ where $\sigma : [n] \to [n]$ is permutation with $\sigma(i) = n - i + 1$. □

Corollary 2.2. *For any random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$, the distributions of S_k and $S_n - S_{n-k}$ are identical for any $k \in [n]$.*

Proof. Using duality principle, we can write the following equality for any $x \in \mathbb{R}$ and step $k \in [n]$

$$P\{S_k \leqslant x\} = P\left\{\sum_{i=1}^{k} X_i \leqslant x\right\} = P\left\{\sum_{i=1}^{k} X_{n-i+1} \leqslant x\right\} = P\left\{\sum_{i=n-k+1}^{n} X_i \leqslant x\right\} = P\{S_n - S_{n-k} \leqslant x\}.$$

□

Corollary 2.3. *For any random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$, the joint distributions of finite sequence $(S_1, \dots, S_n)$ and $(S_n - S_{n-1}, \dots, S_n)$ are identical for any finite $n \in \mathbb{N}$.*

Proof. Since X is *i.i.d.* , it is exchangeable. We take an n-permutation σ such that $\sigma(i) = n - i + 1$ for $i \in [n]$. Then, we observe that the second sequence is $(X_{\sigma(1)}, X_{\sigma(1)} + X_{\sigma(2)}, \dots, \sum_{i=1}^{n} X_{\sigma(i)})$, which is identically distributed to first sequence $(S_1, S_2, \dots, S_n)$. □

Lemma 2.4. *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with an* i.i.d. *step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ having positive mean and natural filtration $\mathcal{F}_\bullet$. We define a discrete process $T : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$, where $T_0 \triangleq 0$ and for each $k \in \mathbb{Z}_+$*

$$T_{k+1} \triangleq \inf\{n > T_k : S_n \leqslant S_{T_k}\} = T_k + \inf\{n \in \mathbb{N} : S_{T_k+n} \leqslant S_{T_k}\}. \tag{1}$$

$P\{T_1 = \infty\} < 1$, and for the associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$, we have $\mathbb{E}N_\infty < \infty$.

Proof. The random time T_k is adapted to the natural filtration of step-size sequence X, for each $k \in \mathbb{N}$. We observe that T_k corresponds to the instant when the random walk S hits kth low. From the strong Markov property for *i.i.d.* sequences, it follows that the distribution of $(X_{T_k+1}, \dots, X_{T_k+n})$ is identical to that of $(X_1, \dots, X_n)$ for any finite $n \in \mathbb{N}$, and independent of $\mathcal{F}_{T_k}$. Therefore, $S_{T_k+n} - S_{T_k}$ has identical distribution to S_n, and is independent of $\mathcal{F}_{T_k}$. Since we can write the difference $T_{k+1} - T_k = \inf\{n \in \mathbb{N} : \sum_{i=1}^{n} X_{T_k+i} \leqslant 0\}$, it follows that T_{k+1} is a random time, and $T_{k+1} - T_k$ is independent of $\mathcal{F}_{T_k}$ and distributed identically to T_1. Therefore, the sequence $(T_k - T_{k-1} : k \in \mathbb{N})$ is *i.i.d.* and $T : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ is a renewal sequence. Associated with renewal sequence T, we can define the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ as $N_n \triangleq \sum_{k \in \mathbb{N}} \mathbb{1}_{\{T_k \leqslant n\}}$ for $n \in \mathbb{N}$. It follows that $\mathbb{E}N_\infty = \sum_{k \in \mathbb{N}} P\{T_k < \infty\}$. We observe that

$$\{T_k < \infty\} = \cap_{j=1}^{k} \{T_j - T_{j-1} < \infty\}.$$

From the L^1 strong law of large numbers, we have $\lim_n \frac{S_n}{n} = \mathbb{E}X_1 > 0$. Thus, $P(\limsup_n \{S_n \leqslant 0\}) = 0$ and for Markov process S, the set of states $\mathbb{R}_-$ is transient. That is, $P\{T_1 < \infty\} < 1$. Since T is a renewal sequence, we get $\mathbb{E}N_\infty = \sum_{k\in\mathbb{N}} P\{T_1 < \infty\}^k = \frac{P\{T_1<\infty\}}{P\{T_1=\infty\}}$ and the result follows. □

Proposition 2.5. *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with an* i.i.d. *step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ having positive mean. Let $\mathcal{F}_\bullet$ be the natural filtration associated with X. The first hitting time of the random walk S to set of positive real numbers, $\tau \triangleq \inf\{n \in \mathbb{N} : S_n > 0\}$, has finite mean. That is, $\mathbb{E}\tau < \infty$.*

Proof. From the definition of random time τ and duality principle, we can write

$$P\{\tau > n\} = P(\cap_{k=1}^n \{S_k \leqslant 0\}) = P(\cap_{k=1}^n \{S_n \leqslant S_{n-k}\}) = P\{S_n \leqslant \min\{0, S_1, \dots, S_{n-1}\}\}.$$

Consider the discrete process $T : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ defined in (1), and observe that $\{\tau > n\} = \cup_{k\in\mathbb{N}} \{T_k = n\}$. Therefore, we can write the mean of stopping time τ as

$$\mathbb{E}\tau = 1 + \sum_{n\in\mathbb{N}} P\{\tau > n\} = 1 + \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} P\{T_k = n\} = 1 + \mathbb{E}\sum_{k\in\mathbb{N}} \mathbb{1}_{\{T_k<\infty\}} = 1 + \mathbb{E}N_\infty.$$

The result follows from finiteness of $\mathbb{E}N_\infty$. □

Definition 2.6. Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with $S_0 \triangleq 0$. The number of distinct values of $(S_0, \cdots, S_n)$ is called **range**, denoted by $R_n \triangleq \cup_{k=0}^n \{S_k\}$. We define the first hitting time of random walk S to $x \in \mathbb{R}$ as the stopping time

$$\tau_x \triangleq \inf\{n \in \mathbb{N} : S_n = x\}.$$

Proposition 2.7. *For a simple random walk, $\lim_{n\in\mathbb{N}} \frac{\mathbb{E}R_n}{n} = P\{\tau_0 = \infty\}$.*

Proof. We can define indicator function for S_k being a distinct number from $S_0, \dots, S_{k-1}$, as $I_k \triangleq \prod_{i=0}^{k-1} \mathbb{1}_{\{S_k \neq S_i\}}$. Then, we can write range R_n in terms of indicator I_k as $R_n = 1 + \sum_{k=1}^n I_k$. From the duality principle

$$P(\cap_{i=1}^k \{S_k \neq S_{k-i}\}) = P(\cap_{i=1}^k \{S_i \neq 0\}) = P\{\tau_0 > k\}, \quad k \in \mathbb{N}.$$

Therefore, $\mathbb{E}R_n = \sum_{k=0}^n P\{\tau_0 > k\}$, and the result follows from Cesàro mean. □

2.1 Simple random walk

Theorem 2.8 (range). *For a simple random walk with $\mathbb{E}X_1 = 2p - 1$, $\lim_{n\in\mathbb{N}} \frac{\mathbb{E}R_n}{n} = 2(p \vee (1-p)) - 1$.*

Proof. When $p = 1 - p$, this random walk is recurrent and thus from the Proposition 2.7, we have

$$\lim_{n\in\mathbb{N}} \frac{\mathbb{E}R_n}{n} = P\{\tau_0 = \infty\} = 0 = 2(p \vee (1-p)) - 1.$$

For $p > (1-p)$, we have $\mathbb{E}X_1 = 2p - 1 > 0$, and therefore $S_n \to \infty$ a.s., and hence

$$P(\{\tau_0 < \infty\} \mid \{X_1 = -1\}) = 1.$$

We define conditional probability $\alpha \triangleq P(\{\tau_0 < \infty\} \mid \{X_1 = 1\})$, and write unconditioned probability of return of random walk to 0 as

$$P\{\tau_0 < \infty\} = \alpha p + (1-p).$$

Since $\tau_0 = 2$ when $S_2 = 0$, we have $P(\{\tau_0 < \infty\} \mid \{S_2 = 0\}) = 1$. From the law of total probability and definition of conditional probability, we get

$$\alpha = P(\{\tau_0 < \infty, X_2 = 1\} \mid \{S_1 = 1\}) + P(\{\tau_0 < \infty, X_2 = -1\} \mid \{S_1 = 1\}) = pP(\{\tau_0 < \infty\} \mid \{S_2 = 2\}) + (1-p).$$

From the Markov property and homogeneity of random walk process, it follows that

$$\begin{aligned} P(\{\tau_0 < \infty\} \mid \{S_2 = 2\}) &= \frac{P\{\tau_0 < \infty, S_2 = 2\}}{P\{S_2 = 2\}} = \frac{P\{\tau_0 < \infty, \tau_1 < \infty, S_2 = 2\}}{P\{S_2 = 2\}} \\ &= P(\{\tau_0 < \infty\} \mid \{\tau_1 < \infty\}) P(\{\tau_1 < \infty\} \mid \{S_2 = 2\}) = \alpha^2. \end{aligned}$$

We conclude $\alpha = \alpha^2 p + 1 - p$, and since $\alpha < 1$ due to transience, we get $\alpha = \frac{1-p}{p}$, and hence the result follows. We can show similarly for the case when $p < 1 - p$. □