Lecture-30: GI/GI/1 Queues

1 GI/GI/1 Queueing Model

Definition 1.1 ($GI/GI/1$ queue). Consider a single server queue with infinite buffer size and FCFS service discipline. We denote the random *i.i.d.* inter-arrival sequence by $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with an arbitrary common distribution $F : \mathbb{R}_+ \to [0,1]$. The random *i.i.d.* service time sequence is denoted by $Y : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with an arbitrary common distribution $G : \mathbb{R}_+ \to [0,1]$. For this $GI/GI/1$ queue, we associate a random walk sequence $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with *i.i.d.* step-size sequence $U : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined as $U_n \triangleq Y_{n-1} - X_n$ for all $n \in \mathbb{N}$. We also define $M : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ as $M_n \triangleq \max\{S_0, \dots, S_n\}$ for all $n \in \mathbb{N}$.

Proposition 1.2 (Lindley's equation). *If we denote the waiting time before service for customer n in the queue by W_n, then we have*

$$W_n = (W_{n-1} + Y_{n-1} - X_n) \vee 0, \quad n \in \mathbb{N}.$$

We denote $W_0 = Y_0 = 0$, and the customer 1 arrives at time X_1.

Proposition 1.3. *Let $W : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be the random waiting time sequence for customers in a GI/GI/1 queue with associated random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$. Then, we have for any $c \geqslant 0$*

$$P\{W_n \geqslant c\} = P\left(\cup_{k \in [n]} \{S_k \geqslant c\}\right). \tag{1}$$

Proof. From the Lindley's recursion for waiting times and the definition of the associated random walk, we get $W_n = \max\{0, W_{n-1} + U_n\}$. Iterating the above relation with $W_1 = 0$, and using the definition of random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ yields

$$W_n = \max\{0, U_n + \max\{0, W_{n-2} + U_{n-1}\}\} = \max\{0, U_n, U_n + U_{n-1} + W_{n-2}\} = \max\{0, S_n - S_{n-1}, \dots, S_n\}.$$

Using the duality principle for exchangeable random sequence U, we get $W_n = M_n$ in distribution. □

Corollary 1.4. *If $\mathbb{E}U_n \geqslant 0$, then we have $P\{W_\infty \geqslant c\} \triangleq \lim_{n \in \mathbb{N}} P\{W_n \geqslant c\} = 1$ for all $c \in \mathbb{R}$.*

Proof. It follows from Proposition 1.3 that $P\{W_n \geqslant c\}$ is non-decreasing in n. Hence, by monotone convergence theorem, the limit exists and is denoted by $P\{W_\infty \geqslant c\} \triangleq \lim_{n \in \mathbb{N}} P\{W_n \geqslant c\}$. Therefore, by continuity of probability and Eq. (1), we have

$$P\{W_\infty \geqslant c\} = P\{S_n \geqslant c \text{ for some } n\}. \tag{2}$$

If $\mathbb{E}U_n = 0$, then the random walk is recurrent, and every state is almost surely reachable. If $\mathbb{E}U_n > 0$, then the random walk S will converge almost surely to positive infinity, from the L^1 strong law of large numbers. □

Remark 1. It follows from this corollary, that the stability condition $\mathbb{E}U_n < 0$ or $\mathbb{E}Y_{n-1} < \mathbb{E}X_n$ is necessary for the existence of a stationary distribution.

Proposition 1.5 (Spitzer's Identity). *Let $M_n \triangleq \max\{0, S_1, S_2, \dots, S_n\}$ for all $n \in \mathbb{N}$, then $\mathbb{E}M_n = \sum_{k=1}^n \frac{1}{k}\mathbb{E}S_k^+$.*

Proof. We can write $M_n = \mathbb{1}_{\{S_n > 0\}} M_n + \mathbb{1}_{\{S_n \leqslant 0\}} M_n$. If $S_n \leqslant 0$, then $M_n = M_{n-1}$. That is, $\mathbb{1}_{\{S_n \leqslant 0\}} M_n = \mathbb{1}_{\{S_n \leqslant 0\}} M_{n-1}$. If $S_n > 0$, then $M_n = \max\{S_1, \dots, S_n\}$. That is,

$$\mathbb{1}_{\{S_n > 0\}} M_n = \mathbb{1}_{\{S_n > 0\}} \max_{i \in [n]} S_i = \mathbb{1}_{\{S_n > 0\}} (U_1 + \max\{0, S_2 - S_1, \dots, S_n - S_1\}).$$

Hence, taking expectation and using exchangeability of the *i.i.d.* sequence U, we get

$$\mathbb{E}[M_n \mathbb{1}_{\{S_n > 0\}}] = \mathbb{E}[U_1 \mathbb{1}_{\{S_n > 0\}}] + \mathbb{E}[M_{n-1} \mathbb{1}_{\{S_n > 0\}}].$$

Since U is an *i.i.d.* sequence and $S_n = \sum_{i=1}^n U_i$, the tuple (U_i, S_n) has an identical joint distribution for all $i \in [n]$. We observe that $M_1 = S_1^+$, and the result follows from

$$\frac{1}{n}\mathbb{E}S_n^+ = \frac{1}{n}\mathbb{E}[S_n \mathbb{1}_{\{S_n>0\}}] = \frac{1}{n}\mathbb{E}\sum_{i=1}^n U_i \mathbb{1}_{\{S_n>0\}} = \mathbb{E}[U_1 \mathbb{1}_{\{S_n>0\}}] = \mathbb{E}M_n - \mathbb{E}M_{n-1}.$$

□

Remark 2. Since $W_n = M_n$ in distribution, we have $\mathbb{E}[W_n] = \mathbb{E}[M_n] = \sum_{k=1}^n \frac{1}{k}\mathbb{E}[S_k^+]$.

2 Martingales for Random Walks

Proposition 2.1. *Consider an* i.i.d. *step-size sequence* $X : \Omega \to \mathbb{Z}^{\mathbb{N}}$ *such that* $|X_n| \leqslant M \in \mathbb{Z}_+$. *A random walk* $S : \Omega \to \mathbb{Z}^{\mathbb{N}}$ *with the step size sequence* X *is a recurrent Markov chain iff* $\mathbb{E}X_n = 0$.

Proof. If $\mathbb{E}X_n \neq 0$, the random walk is clearly transient since, it will diverge to $\pm\infty$ depending on the sign of $\mathbb{E}X_n$.

Conversely, if $\mathbb{E}X_n = 0$, then the random walk S is a martingale adapted to natural filtration $\mathcal{F}_\bullet$ of the step-size sequence. Assume that the random walk starts at state $S_0 = x \in \mathbb{Z}_+$. We define sets

$$A \triangleq \{-M, -M+1, \cdots, -2, -1\}, \qquad A_y \triangleq \{y+1, \dots, y+M\}, \quad y > x.$$

Let $\tau \triangleq \inf\{n \in \mathbb{N} : S_n \in A \cup A_y\}$ denote the first hitting time by the random walk S to either A or A_y. It follows that τ is a stopping time adapted to $\mathcal{F}_\bullet$. Further, $\sup_{n\in\mathbb{N}} |S_{\tau\wedge n}| \leqslant y + M$. From the optional stopping theorem, we have $\mathbb{E}S_\tau = \mathbb{E}S_0 = x$. Thus, we have

$$x = \mathbb{E}_x S_\tau = \mathbb{E}_x[S_\tau \mathbb{1}_{\{S_\tau\in A\}} + S_\tau \mathbb{1}_{\{S_\tau\in A_y\}}] \geqslant -MP_x\{S_\tau \in A\} + y(1 - P_x\{S_\tau \in A\}).$$

Rearranging the above equation, we get a bound on probability of random walk S hitting A over A_y as

$$P_x\{S_n \in A \text{ for some } n\} \geqslant P_x\{S_\tau \in A\} \geqslant \frac{y-x}{y+M}.$$

Since the choice of $y \in \mathbb{Z}_+$ was arbitrary, taking limit $y \to \infty$, we see that for any $x \in \mathbb{Z}_+$, we have $P_x\{S_n \in A \text{ for some } n\} = 1$. Similarly taking $B \triangleq \{1, 2, \cdots, M\}$, we can show that $P_x\{S_n \in B \text{ for some } n\} = 1$ for any $x \leqslant 0$. Result follows from combining the above two arguments to see that for any $x \in \mathbb{Z}$

$$P_x\{S_n \in A \cup B \text{ for some } n\} = 1.$$

□

Proposition 2.2. *Consider a random walk* $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ *with* i.i.d. *step-size sequence* $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ *with common mean* $\mathbb{E}[X_1] \neq 0$. *For* $a, b > 0$, *we define the hitting time of the walk* S *exceeding a positive threshold* a *or going below a negative threshold* $-b$ *as*

$$\tau \triangleq \{n \in \mathbb{N} : S_n \geqslant a \text{ or } S_n \leqslant -b\}.$$

Let P_a *denote the probability that the walk hits a value greater than* a *before it hits a value less than* $-b$. *That is,* $P_a \triangleq P\{S_\tau \geqslant a\}$. *Then, for* $\theta \neq 0$ *such that* $\mathbb{E}e^{\theta X_1} = 1$, *we have* $P_a \approx \frac{1-e^{-\theta b}}{e^{\theta a}-e^{-\theta b}}$. *The above approximation is an equality when step size is unity and* a *and* b *are integer valued.*

Proof. For any $a, b > 0$, we can define stopping times

$$\tau_a = \inf\{n \in \mathbb{N} : S_n \geqslant a\}, \qquad \tau_{-b} = \inf\{n \in \mathbb{N} : S_n \leqslant -b\}.$$

Then, $\tau = \tau_a \wedge \tau_{-b}$, and we are interested in computing the probability $P_a = P\{\tau_a < \tau_{-b}\}$. We define a random sequence $Z : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ such that $Z_n \triangleq e^{\theta S_n}$ for all $n \in \mathbb{N}$, where $\mathbb{E}e^{\theta X_1} = 1$. Hence, it follows that Z is a martingale with unit mean. We observe that $\sup_{n\in\mathbb{N}} |Z_{\tau\wedge n}| \leqslant e^{\theta a} \vee e^{-\theta b}$. From the optional stopping theorem, we get $\mathbb{E}e^{\theta S_\tau} = 1$. Thus, we get

$$1 = \mathbb{E}[e^{\theta S_\tau}\mathbb{1}_{\{\tau_a<\tau_{-b}\}}] + \mathbb{E}[e^{\theta S_\tau}\mathbb{1}_{\{\tau_a>\tau_{-b}\}}].$$

We can approximate $e^{\theta S_\tau}\mathbb{1}_{\{\tau_a<\tau_{-b}\}}$ by $e^{\theta a}\mathbb{1}_{\{\tau_a<\tau_{-b}\}}$ and $e^{\theta S_\tau}\mathbb{1}_{\{\tau_a>\tau_{-b}\}}$ by $e^{-\theta b}\mathbb{1}_{\{\tau_a>\tau_{-b}\}}$, by neglecting the overshoots past the thresholds a and $-b$. Therefore, we have

$$1 \approx e^{\theta a}P_a + e^{-\theta b}(1 - P_a).$$

□

Corollary 2.3. *Let* $\tau \triangleq \tau_a \wedge \tau_{-b}$ *and* $P_a \triangleq P\{\tau_a < \tau_{-b}\}$, *then* $\mathbb{E}\tau \approx \frac{aP_a - b(1-P_a)}{\mathbb{E}X_1}$.