1. Consider a single link network shared by the set $\mathcal{S} \triangleq \{0,1,2\}$ of three users. We denote the source rate vector $x \in \mathbb{R}_+^{\mathcal{S}}$. Obtain the optimal resource allocation, when the link has unit capacity and the utility functions associated with the three users are

$$U_0(x_0) = \ln x_0, \qquad U_1(x_1) = 3\ln x_1, \qquad U_2(x_2) = 4\ln x_2.$$

2. Consider a two-link, three-source network shown in Figure 1, where link A has a capacity of 2 packets/time slot and link B has a capacity of 1 packet/time slot. The route of source 0 consists of both links A and B, the route of source 1 consists of only link A, and route of source 2 consists of only link B. Compute the resource allocations under the proportional fairness, minimum potential delay fairness, and max-min fairness.

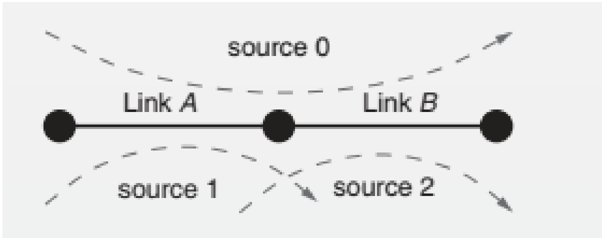

Figure 1: A two link and three source network.

3. Consider a two link and three source network shown in Figure 1, where link A has a capacity of 10 packets/time slot and link B has a capacity of 5 packet/time slot. The route of source 0 consists of both links A and B, the route of source 1 consists of only link A, and the route of source 2 consists of only link B. Compute α-fair rate allocation for $\alpha \in \{1,2,5\}$.

4. Consider a network with the set of users $\mathcal{S}$, the set of links $\mathcal{L}$, and a fixed routing matrix $R \in \{0,1\}^{\mathcal{L}\times\mathcal{S}}$. We denote the source rate vector by $x \in \mathbb{R}_+^{\mathcal{S}}$, the link capacity vector by $c \in \mathbb{R}_+^{\mathcal{L}}$, and link load vector by $y = Rx \in \mathbb{R}_+^{\mathcal{L}}$. A link ℓ is called a *bottleneck link* for user $r \in \mathcal{S}$ if

   (a) $R_{\ell,r} = 1$,
   (b) $y_\ell = c_\ell$, and
   (c) $x_s \leqslant x_r$ for all $\{s \in \mathcal{S} : R_{\ell,s} = 1\}$.

   That is, if link ℓ is in the route of user r, is fully utilized, and user r has the highest transmission rate among all users using link ℓ. Show that x is a max-min fair rate allocation if and only if every source has at least one bottleneck link.

5. Consider a network with set of links $\mathcal{L}$, shared by a set of users $\mathcal{S}$, with a fixed routing matrix $R \in \{0,1\}^{\mathcal{L}\times\mathcal{S}}$. We denote the rate and aggregate price vectors for sources by $x,q \in \mathbb{R}_+^{\mathcal{S}}$ respectively, and capacity, loads, price, and scaling vectors for links by $c,y,p,h \in \mathbb{R}_+^{\mathcal{L}}$ respectively. Recall that $y \triangleq Rx, q \triangleq R^\top p$, and the dual algorithm for rate and price updates

$$x_r \triangleq U_r'^{-1}(q_r), \qquad \dot{p}_\ell \triangleq h_\ell(y_\ell - c_\ell)^+_{p_\ell}.$$

   Show that the dual algorithm is globally asymptotically stable when the routing matrix R has full row rank, i.e., given q, there exists a unique p that satisfies $q = R^\top p$.