Lecture-23: Stability of dynamic systems

1 Dynamic system

Definition 1.1 (Autonomous dynamic system). For a continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$, we consider an *autonomous dynamic system* $x : \mathbb{R}_+ \to \mathbb{R}^n$ defined by the following differential equation for all $t \in \mathbb{R}_+$,

$$\dot{x}(t) = \frac{d}{dt}x(t) = f(x(t)).$$

The time variable t is omitted when no confusion is caused. We assume that $x(0)$ is given and f satisfies other appropriate conditions to ensure that the differential equation has a unique solution

$$x(t) = x(0) + \int_0^t f(x(s))ds, \text{ for } t \in \mathbb{R}_+.$$

Definition 1.2 (Equilibrium point). A point $x_e \in \mathbb{R}^n$ is called an *equilibrium point* of autonomous dynamic system $x : \mathbb{R}_+ \to \mathbb{R}^n$ defined in Definition 1.1 if $f(x_e) = 0$. The set of equilibrium points is denoted by $A_e \triangleq \{x \in \mathbb{R}^n : f(x) = 0\}$. We assume that x_e is the unique equilibrium point of this autonomous dynamic system.

Lemma 1.3. *Consider an autonomous dynamic system $x : \mathbb{R}_+ \to \mathbb{R}^n$ defined in Definition 1.1. If $x(t) = x_e$ for some $t \in \mathbb{R}_+$, then $x(s) = x_e$ for all $s > t$.*

Proof. Let $x(t) = x_e$ for some $t \in \mathbb{R}_+$ and we define $u \triangleq \inf\{s > t : x(s) \neq x_e\}$, then u is a point of discontinuity whereas x is continuous everywhere in t. This implies that $u = \infty$, and we are done. □

Corollary 1.4. *Consider the autonomous dynamic system $x : \mathbb{R}_+ \to \mathbb{R}^n$ defined in Definition 1.1. If $x(t)$ does not converge to the equilibrium point x_e for large t, then $x(t) \neq x_e$ for any $t \in \mathbb{R}_+$.*

Proof. If $x(t) = x_e$ for some $t \in \mathbb{R}_+$, then from Lemma 1.3 we have $x(s) = x_e$ for all $s \geqslant t$. This implies that $\lim_{t\to\infty} x(t) = x_e$ and contradicts the assumption that $x(t)$ does not converge to x_e. □

Definition 1.5 (Potential function). A map $V : \mathbb{R}^n \to \mathbb{R}$ is called a *potential function*, if it is differentiable and satisfies the radial unboundedness property $\lim_{\|x\|\to\infty} V(x) = \infty$.

Lemma 1.6. *Let $c \in \mathbb{R}$ and $V : \mathbb{R}^n \to \mathbb{R}$ a potential function, then the set $A_c \triangleq \{x \in \mathbb{R}^n : V(x) \leqslant c\}$ is bounded.*

Proof. If set A_c is unbounded, then we can find a sequence $y \in A_c^{\mathbb{N}}$ such that $\lim_{m\to\infty} \|y_m\| = \infty$. It follows that $V(y_m) \leqslant c$ for all $m \in \mathbb{N}$, and hence $\lim_{m\to\infty} V(y_m) \leqslant c$. However this contradicts the radial unboundedness property of potential function V. □

Lemma 1.7. *For any potential function $V : \mathbb{R}^n \to \mathbb{R}$ and an autonomous dynamic system $x : \mathbb{R}_+ \to \mathbb{R}^n$ defined in Definition 1.1, the time derivative of potential function is denoted by $\dot{V}(x) \triangleq \frac{d}{dt}V(x(t))$, and given by*

$$\dot{V}(x) = \langle \nabla V(x), \dot{x} \rangle = \langle \nabla V(x), f(x) \rangle.$$

Proof. It follows from substituting the definition of autonomous dynamic system in the time derivative of potential function, and applying chain rule. □

Theorem 1.8 (Lyapunov boundedness). *Consider an autonomous dynamic system x defined in Definition 1.1 and an associated potential function $V : \mathbb{R}^n \to \mathbb{R}$ defined in Definition 1.5. If $\dot{V}(x) \leqslant 0$ for all x, then there exists a constant $B > 0$ such that $\|x(t)\| \leqslant B$ for all times t.*

Proof. From Lemma (1.6), we get that $A_c \triangleq \{x \in \mathbb{R}^n : V(x) \leqslant c\}$ is a bounded set for any finite $c \in \mathbb{R}$. Since $\dot{V}(x) < 0$ for all x, we get that at any time $t \in \mathbb{R}_+$, we have

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))ds \leqslant V(x(0)).$$

Taking $c \triangleq V(x(0))$, we get that $x(t) \in A_c$ for all $t \in \mathbb{R}_+$. The result follows by taking $B \triangleq \sup\{\|x\| : x \in A_c\}$. □

Definition 1.9 (Globally asymptotically stable). An equilibrium point $x_e \in A_e$ is said to be a *globally asymptotically stable* if $\lim_{t\to\infty} x(t) = x_e$ for any $x(0) \in \mathbb{R}^n$.

Example 1.10 (Not globally asymptotically stable). Consider an autonomous dynamic system $x : \mathbb{R}_+ \to \mathbb{R}$ defined as $\dot{x} = e^x - 1$. We observe that it has a unique rest point $x_e = 0$. However, if $x(0) > 0$ then we observe that $\lim_{t\to\infty} x(t) = \infty$ and if $x(0) < 0$ then $\lim_{t\to\infty} x(t) = -\infty$.

Theorem 1.11 (Lyapunov global asymptotic stability). *Consider an autonomous dynamic system x defined in Definition 1.1 and an associated potential function $V : \mathbb{R}^n \to \mathbb{R}$ defined in Definition 1.5 that satisfies the following properties.*
(a) V is differentiable with continuous first derivatives,
(b) $V(x) \geqslant 0$ for all $x \in \mathbb{R}^n$ with equality iff $x = x_e$, and
(c) $\dot{V}(x) \leqslant 0$ for any $x \in \mathbb{R}^n$ with equality iff $x = x_e$.
Then the equilibrium point x_e is globally asymptotically stable.

Proof. We prove this theorem by contradiction, and assume that $x(t)$ doesn't converge to x_e for large t. Therefore, $x(t) \neq x_e$ for any $t \in \mathbb{R}_+$ from Corollary 1.4. Consequently, $\dot{V}(x(t)) < 0$ and $V(x(t)) > 0$ for all times $t \in \mathbb{R}_+$, and hence $V(x(t))$ is decreasing in time t and lower bounded by 0. Hence, $\epsilon \triangleq \lim_{t\to\infty} V(x(t)) \geqslant 0$ exists. Since V and x are continuous, we get that $\lim_{t\to\infty} V(x(t)) = V(\lim_{t\to\infty} x(t)) \neq V(x_e) = 0$. Therefore $\epsilon > 0$, and we define the set

$$\mathcal{C} \triangleq \{x \in \mathbb{R}^n : \epsilon \leqslant V(x) \leqslant V(x(0))\} = V^{-1}[\epsilon, V(x(0))] \subseteq V^{-1}(-\infty, V(x(0))].$$

We observe that $x(t) \in \mathcal{C}$ for all $t \in \mathbb{R}_+$. From Lemma 1.6 for potential function, the set $V^{-1}(-\infty, V(x(0))]$ is bounded and hence so is $\mathcal{C}$. Further, $\mathcal{C}$ is closed since the map $x \mapsto V(x)$ is continuous and $[\epsilon, V(x(0))]$ is closed. Since $\mathcal{C}$ is closed and bounded, it is a compact set and we define

$$a \triangleq \sup\{\dot{V}(x) : x \in \mathcal{C}\} = \sup \dot{V}^{-1}(\mathcal{C}) \leqslant 0.$$

Since $\mathcal{C}$ is compact and the map $x \mapsto \dot{V}(x)$ is continuous, it follows that $\dot{V}^{-1}(\mathcal{C})$ is compact and $\sup \dot{V}^{-1}(\mathcal{C}) = \max \dot{V}^{-1}(\mathcal{C})$ is finite and belongs to $\dot{V}^{-1}(\mathcal{C})$. Since $x_e \notin \mathcal{C}$ and hence $0 \notin \dot{V}^{-1}(\mathcal{C})$, it follows that $a < 0$ is finite. Hence, we can write

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))ds \leqslant V(x(0)) + at.$$

This implies that $V(x(t)) = 0$ and $x(t) = x_e$ for all $t \geqslant -\frac{1}{a}V(x(0))$. This contradicts the assumption that $x(t)$ does not converge to x_e. □

Remark 1. The Lyapunov global asymptotic stability theorem requires that $\dot{V}(x) \neq 0$ for any $x \neq x_e$. In the case $\dot{V}(x) = 0$ for some $x \neq x_e$, global asymptotic stability can be studied using Lasalle's invariance principle.

Theorem 1.12 (Lasalle's invariance principle). *Consider an autonomous dynamic system x defined in Definition 1.1 and an associated potential function $V : \mathbb{R}^n \to \mathbb{R}$ defined in Definition 1.5 that satisfies the following properties.*
(a) V is differentiable with continuous first derivatives,
(b) $V(x) \geqslant 0$ for all $x \in \mathbb{R}^n$ with equality iff $x = x_e$,

(c) $\dot{V}(x) \leqslant 0$ *for all* $x \in \mathbb{R}^n$*, and*
(d) the only trajectory $x(t)$ *that satisfies* $\dot{x}(t) = f(x(t))$ *and* $\dot{V}(x(t)) = 0$ *for all* $t \in \mathbb{R}_+$*, is* $x(t) = x_e$ *for all* $t \in \mathbb{R}_+$*.*
Then the equilibrium point x_e *is globally asymptotically stable.*

Proof. □