Lecture-02: Embedded Markov Chain and Sojourn Times

1 State Evolution

For a homogeneous Markov process $X : \Omega \to \mathcal{X}_+^{\mathbb{R}}$ on countable state space $\mathcal{X} \subseteq \mathbb{R}$ with sample paths that are right continuous with left limits (rcll), we wish to characterize the transition kernel $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X}\times\mathcal{X}}$, where $P_{xy}(t) \triangleq P(\{X_{s+t} = y\} \mid \{X_s = x\})$ for all $s, t \in \mathbb{R}_+$. To this end, we define the sojourn time in any state, the jump times, and the jump transition probabilities.

1.1 Transition instants and counting process

Definition 1.1. Let $S_0 \triangleq 0$. The nth **jump time** of a right continuous countable state stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ is defined inductively as $S_n \triangleq \inf\left\{t > S_{n-1} : X_t \neq X_{S_{n-1}}\right\}$.

Definition 1.2. The counting process associated with jump times sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is denoted by $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$, where the number of jumps in duration $(0,t]$ is denoted by $N_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$.

Lemma 1.3. *Each term of the jump time sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is adapted to the natural filtration $\mathcal{F}_\bullet$ of the process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$.*

Proof. Since X is rcll, it is progressively measurable, and hence the event $\{S_n \leqslant t\} \in \mathcal{F}_t$. □

Definition 1.4 (Age of a state). For the counting process N associated with the CTMC X, we can define age process $A : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ where the age of last transition at time t is denoted by $A_t \triangleq t - S_{N_t}$ for all times $t \in \mathbb{R}_+$. For the CTMC X, we can write the age of last transition at time $t \in \mathbb{R}_+$ as

$$A_t \triangleq \inf\left\{s > 0 : X_{t-s} \neq X_t\right\}.$$

Definition 1.5 (Excess time in a state). For the counting process N associated with the CTMC X, we can define the excess time process $Y : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ where $Y_t \triangleq S_{N_t+1} - t$ is the time until next transition at time t. We can write the excess time at time $t \in \mathbb{R}_+$ for the CTMC X as

$$Y_t \triangleq \inf\left\{s > 0 : X_{t+s} \neq X_t\right\}.$$

Remark 1. For a homogeneous CTMC X, the distribution of excess time Y_t conditioned on the current state X_t, doesn't depend on time t. Hence, we can define the following conditional complementary distribution of excess time as $\bar{F}_x(u) \triangleq P(\{Y_t > u\} \mid \{X_t = x\}) = P_x\left\{Y_0 > u\right\}$.

Lemma 1.6. *For a homogeneous CTMC X, there exists a positive sequence $\nu \in \mathbb{R}_+^{\mathcal{X}}$, such that*

$$\bar{F}_x(u) \triangleq P(\{Y_t > u\} \mid \{X_t = x\}) = e^{-u\nu_x}, \quad x \in \mathcal{X}.$$

Proof. We fix a state $x \in \mathcal{X}$, and observe that the function $\bar{F}_x : \mathbb{R}_+ \to [0,1]$ is non-negative, non-increasing, and right-continuous. Using the Markov property and the time-homogeneity, we can show that $\bar{F}_x$ satisfies the semigroup property. In particular,

$$\bar{F}_x(u+v) = P(\{Y_t > u+v\} \mid \{X_t = x\}) = P(\{Y_t > u, Y_{t+u} > v\} \mid \{X_t = x\}) = \bar{F}_x(u)\bar{F}_x(v).$$

The only continuous function $\bar{F}_x : \mathbb{R}_+ \to [0,1]$ that satisfies the semigroup property is an exponential function with a negative exponent. □

Example 1.7 (Poisson process). Consider the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ for a Poisson point process with homogeneous rate λ. Using the stationary independent increment property, we have for all $u \geqslant 0$

$$\bar{F}_i(u) = P(\{Y_t > u\} \mid \{N_t = i\}) = P(\{N_{t+u} = i\} \mid \{N_t = i\}) = P\left\{N_{t+u} - N_t = 0\right\} = P\left\{Y_t > u\right\} = e^{-\lambda u}.$$

1.2 Sojourn time in a state

Definition 1.8. The **jump process** $Z : \Omega \to \mathcal{X}^{\mathbb{N}}$ is a discrete time process, derived from the continuous time stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ by sampling X at the jump times $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. The state of the process X at the nth jump time S_n is the nth state $Z_n \triangleq X_{S_n}$ of the jump process Z.

Definition 1.9. The **sojourn time** in the state Z_{n-1} for the process X is defined as $T_n \triangleq S_n - S_{n-1}$.

Remark 2. From the definition of jump instants, it follows that the history until time t is

$$\mathcal{F}_t = \sigma(S_0, (Z_0, T_1), (Z_1, T_2), \dots, (Z_{N_t}, A_t)).$$

We can verify that $\mathcal{F}_{S_n} = \sigma(S_0, (Z_0, T_1), \dots, (Z_{n-1}, T_n), Z_n)$.

Lemma 1.10. *For a homogeneous CTMC, each sojourn time $T_n : \Omega \to \mathbb{R}_+$ is a continuous memoryless random variable, and the sequence of sojourn times $(T_j : j \geqslant n)$ is independent of the past $\mathcal{F}_{S_{n-1}}$ conditioned on Z_{n-1}.*

Proof. We observe that the sojourn time T_n equals the excess time $Y_{S_{n-1}}$, where the process remains in state $Z_{n-1} = X_{S_{n-1}}$ in the duration $S_{n-1} + [0, T_n)$. Using the strong Markov property, we can write the conditional complementary distribution of T_n given any historical event $H \in \mathcal{F}_{S_{n-1}}$ and $u \geqslant 0$ as

$$P(\{T_n > u\} \mid \{Z_{n-1} = x\} \cap H) = P(\{Y_{S_{n-1}} > u\} \mid \{X_{S_{n-1}} = x\} \cap H) = \exp(-u\nu_x) = \bar{F}_x(u).$$

$\square$

Corollary 1.11. *If $X_n = x$, then the holding time T_{n+1} is an exponential random variable with rate ν_x.*

Definition 1.12. For a homogeneous CTMC X, the exponential rate for the random holding time in a state x is called the **transition rate** out of state x denoted by ν_x.

Definition 1.13. For a CTMC X, a state $x \in \mathcal{X}$ is called **instantaneous** if $\nu_x = \infty$, **stable** if $\nu_x \in (0, \infty)$, and **absorbing** if $\nu_x = 0$.

Remark 3. Transition rate out of a state x is the inverse of mean holding time in this state x, i.e. $\nu_x = \frac{1}{\mathbb{E}_x T_1}$. Therefore, the mean holding time $\mathbb{E}_x T_1$ in state x is ∞ in an absorbing state, zero in an instantaneous state, and almost surely finite and non-zero in a stable state.

Definition 1.14. A homogeneous CTMC with no instantaneous states is called a **pure jump** CTMC.

Definition 1.15. A pure jump CTMC with
(i) all stable states and $\inf_{x \in \mathcal{X}} \nu_x \geqslant \nu > 0$ is called **stable**, and
(ii) $\sup_{x \in \mathcal{X}} \nu_x \leqslant \nu < \infty$ is called **regular**.

Example 1.16 (Non-regular CTMC). For the countable state space $\mathbb{N}$, consider the probability transition matrix P such that $p_{n,n+1} = 1$ and the exponential holding times with rate $\nu_n = n^2$ for each state $n \in \mathbb{N}$. Clearly, $\sup_{n \in \mathbb{N}} \nu_n = \infty$, and hence it is not regular.

Remark 4. Pure jump homogeneous CTMC with finite stable states are stable and regular. We will focus on pure jump homogeneous CTMC over countably infinite states, that are stable and regular.

1.3 Jump process

Proposition 1.17. *For a stable CTMC, the jump times are stopping times.*

Proof. For a stable CTMC X, we let $0 < \nu \leqslant \inf_{x \in \mathcal{X}} \nu_x$. Then, by coupling in Appendix B, we have a sequence of *i.i.d.* random variables $\overline{T} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, such that $T_n \leqslant \overline{T}_n$ almost surely and $\mathbb{E}\overline{T}_n = \frac{1}{\nu}$ for each $n \in \mathbb{N}$. Defining $\overline{S}_n \triangleq \sum_{i=1}^{n} \overline{T}_i$, it follows that $S_n \leqslant \overline{S}_n$ for each $n \in \mathbb{N}$. Since $\overline{S}_n$ is sum of n almost surely finite random variables, it is finite almost surely. It follows that S_n is finite almost surely. $\square$

Proposition 1.18. *For a regular CTMC, N_t is almost surely finite for all finite times $t \in \mathbb{R}_+$.*

Proof. Let X be a regular CTMC and $\sup_{x\in\mathcal{X}} \nu_x \leqslant \nu < \infty$. Then, by coupling in Appendix B, we have a sequence of *i.i.d.* random variables $\underline{T} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, such that $T_n \geqslant \underline{T}_n$ almost surely and $\mathbb{E}\underline{T}_n = \frac{1}{\nu}$ for each $n \in \mathbb{N}$. Defining $\underline{S}_n \triangleq \sum_{i=1}^n \underline{T}_i$ and $\underline{N}_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\underline{S}_n \leqslant t\}}$, it follows that $S_n \geqslant \underline{S}_n$ for each $n \in \mathbb{N}$ and $N_t \leqslant \underline{N}_t$ for all $t \in \mathbb{R}_+$. Since $\underline{N}$ is a Poisson counting process with finite rate ν, it is almost surely finite at all $t \in \mathbb{R}_+$ and the result follows. □

Remark 5. From the strong Markov property and the time-homogeneity of the CTMC X, we see that

$$P(\{Z_n = y\} \mid \{Z_{n-1} = x\}) = P_{xy}(S_{n-1}, S_n) = P_{xy}(0, T_n).$$

Remark 6. From the law of total probability, it follows that for any rcll stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with countable state space $\mathcal{X}$, the sum of jump transition probabilities $\sum_{y\neq x} P_{xy}(S_{n-1}, S_n) = 1$ for all states $X_{S_{n-1}} = x \in \mathcal{X}$.

Lemma 1.19. *For a homogeneous CTMC X, the jump probability from state Z_{n-1} to state Z_n depends solely on Z_{n-1} and is independent of jump instants.*

Proof. Fix states $x, y \in \mathcal{X}$ and a historical event $H \in \mathcal{F}_{S_{n-1}}$. From the definition of conditional probability, we write

$$P(\{T_n > u, Z_n = y\} \mid \{Z_{n-1} = x\}\cap H) = P(\{X_{S_n} = y\} \mid \{T_n > u, X_{S_{n-1}} = x\}\cap H)P(\{T_n > u\} \mid \{Z_{n-1} = x\}\cap H).$$

From the strong Markov property of X, we get $P(\{T_n > u\} \mid \{Z_{n-1} = x\} \cap H) = \bar{F}_x(u)$. We further observe that $\{T_n > u, X_{S_{n-1}} = x\} \cap H = \{X_t = x, t \in S_{n-1} + [0, u]\} \cap H \in \mathcal{F}_{S_{n-1}+u}$. From the definition of excess time, we can write $S_n = S_{n-1} + u + Y_{S_{n-1}+u}$ for any $u \in [0, T_n]$. Further, from the strong Markov and the time-homogeneity of CTMC X, and the memoryless property of excess time Y, we obtain

$$P(\{X_{S_n} = y\} \mid \{T_n > u, X_{S_{n-1}} = x\}\cap H) = P(\left\{X_{S_{n-1}+u+Y_{S_{n-1}+u}} = y\right\} \mid \{X_{S_{n-1}+u} = x\}) = P_{xy}(0, Y_0).$$

This implies that sojourn times and jump instant probabilities are independent. □

Definition 1.20. The jump process Z is also sometimes referred to as the **embedded DTMC** of the pure jump CTMC X. The corresponding **jump transition probabilities** are defined

$$p_{xy} \triangleq P_{xy}(S_{n-1}, S_n) = P(\{X_{S_n} = y\} \mid \{X_{S_{n-1}} = x\}), \quad x, y \in \mathcal{X}.$$

Remark 7. If $\nu_x = 0$, then for any $u \geqslant 0$, we have $P(\{Y_0 > u\} \mid \{X_0 = x\}) = 1$, and hence $S_1 = \infty$ almost surely whenever $X_0 = x$. By convention, we set $p_{xx} = 1$ and $p_{xy} = 0$ for all states $y \neq x$.

Corollary 1.21. *The matrix $p \triangleq (p_{xy} : x, y \in \mathcal{X})$ is stochastic, and if $\nu_x > 0$ then $p_{xx} = 0$.*

Proof. Recall $p_{xy} = P_{xy}(S_1)$. If $\nu_x > 0$, then $\lim_{u\to\infty} P(\{Y_0 > u\} \mid \{X_0 = x\}) = 0$, and hence S_1 is finite almost surely. By definition $X_{S_1} \neq X_0 = x$, and hence $p_{xx} = 0$. □

Proposition 1.22. *Consider a stable CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$. Then for all states $x, y \in \mathcal{X}$ and duration $u \in \mathbb{R}_+$,*

$$P(\{T_{n+1} > u, Z_{n+1} = y\} \mid \{X_0 = x_0, \dots, Z_n = x, S_0 \leqslant s_0, \dots, S_n \leqslant s_n\}) = p_{xy}e^{-u\nu_x}.$$

Proof. The history of the process until stopping time S_n is given by $\mathcal{F}_{S_n} = \sigma(S_0, (Z_0, T_1), \dots, (Z_{n-1}, T_n), Z_n)$. Therefore $H \triangleq \{S_0 \leqslant s_0\} \cap_{i=1}^n \{Z_{i-1} = x_{i-1}, S_i \leqslant s_i\} \in \mathcal{F}_{S_n}$ and $\{Z_n = x\} \cap H \in \mathcal{F}_{S_n}$. Using strong Markov property and time-homogeneity of the CTMC X, we have

$$P(\{T_{n+1} > u, Z_{n+1} = y\} \mid \{Z_n = x\} \cap H) = P_x\{S_1 > u, Z_1 = y\}.$$

The result follows from the previous Lemma 1.19. □

Corollary 1.23. *For a time-homogeneous CTMC, the transition probabilities $(p_{xy} : x, y \in \mathcal{X})$ and holding times $T : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ are independent. The jump process $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ is a homogeneous Markov chain with countable state space $\mathcal{X}$. The holding time sequence $T : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is independent and T_n is distributed exponentially with rate $\nu_{Z_{n-1}}$ for each $n \in \mathbb{N}$.*

Example 1.24 (Poisson process). For a Poisson counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with time-homogeneous rate λ, the countable state space is $\mathbb{Z}_+$, and transition rate $\nu_n = \lambda$ for each state $n \in \mathbb{Z}_+$. It follows from the memoryless property of exponential random variables, that

$$\bar{F}_n(t) = P(\{Y_u > t\} \mid \{N_u = n\}) = P\{S_1 > t\} = e^{-\lambda t}.$$

Further, the embedded Markov chain or the jump process is given by the initial state $N_0 = 0$ and the transition probability matrix $P = (p_{n,m} : n, m \in \mathbb{Z}_+)$ where $p_{n,n+1} = 1$ and $p_{n,m} = 0$ for $m \neq n+1$. This follows from the definition of T_1, since $p_{n,m} = P(\{N_{T_1} = m\} \mid \{N_0 = m\}) = \mathbb{1}_{\{m=n+1\}}$.

Theorem 1.25. *A pure-jump homogeneous CTMC whose embedded DTMC is recurrent is regular.*

Proof. Let $X_0 = x \in \mathcal{X}$ be the initial state. Let $N_x(n) = \sum_{k=1}^{n} \mathbb{1}_{\{Z_k = x\}}$ be the number of visits to a state $x \in \mathcal{X}$ in the first n transitions and T_i^x be the ith sojourn time in the state x. From the recurrence of the embedded chain, the state x occurs infinitely often, i.e. $\lim_{n\in\mathbb{N}} N_x(n) = \infty$ almost surely. It follows that the sojourn time sequence $T^x : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* and exponentially distributed with mean $\mathbb{E}T_i^x = \frac{1}{\nu_x} < \infty$. Since $S_n \geqslant \sum_{i=1}^{N_x(n)} T_i^x$, we get that

$$m_t = \sum_{n\in\mathbb{N}} P\{S_n \leqslant t\} \leqslant \sum_{n\in\mathbb{N}} P\left\{\sum_{i=1}^{N_x(n)} T_i^x \leqslant t\right\} = \nu_x t.$$

It follows that N_t is almost surely finite for any finite time $t \in \mathbb{R}_+$. □

A Exponential random variables

Lemma A.1. *Let X be an exponential random variable, and S be any positive random variable, independent of X. Then, for all $u \geqslant 0$*

$$P(\{X > S + u\} \mid \{X > S\}) = P\{X > u\}.$$

Proof. Let the distribution of S be F and X be memoryless with rate μ. From the definition of conditional probability, we can write $P(\{X > S+u\} \mid \{X > S\}) = \frac{P\{X>S+u\}}{P\{X>S\}}$. Since a probability is an expectation of an indicator, we can write

$$P\{X > S+u\} = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{X>S+u\}} \mid \sigma(S)]] = \mathbb{E}[e^{-\mu(S+u)}] = e^{-\mu u}\mathbb{E}[e^{-\mu S}].$$

It follows that $P(\{X > S+u\} \mid \{X > S\}) = P\{X > u\} = e^{-\mu u}$ for all $u \in \mathbb{R}_+$. □

B Coupling

For a homogeneous regular and stable CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$, we denote the embedded Markov chain by $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ and the independent inter-jump time sequence by $T : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ where T_n is an exponential random variable with rate $\nu_{Z_{n-1}}$ for all $n \in \mathbb{N}$. From the regularity and stability of process X, we have

$$0 < \overline{\nu} \leqslant \inf_{x\in\mathcal{X}} \nu_x \leqslant \sup_{x\in\mathcal{X}} \nu_x \leqslant \underline{\nu} < \infty.$$

Consider an *i.i.d.* uniform random sequence $U : \Omega \to [0,1]^{\mathbb{N}}$ and define three dependent random sequences $\underline{T}, T, \overline{T} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have

$$\overline{T}_n \triangleq -\frac{1}{\overline{\nu}} \log U_n, \qquad \underline{T}_n \triangleq -\frac{1}{\underline{\nu}} \log U_n, \qquad T_n \triangleq -\frac{1}{\nu_{Z_{n-1}}} \log U_n.$$

We observe that $\underline{T}$ and $\overline{T}$ are *i.i.d.* exponential random sequences with rates $\underline{\nu}$ and $\overline{\nu}$ respectively. Further, T is an independent exponential random sequence with the rate $\nu_{Z_{n-1}}$ for T_n. In addition, by construction, we have $\underline{T}_n \leqslant T_n \leqslant \overline{T}_n$ for each $n \in \mathbb{N}$.