Lecture-07: Mean-field model

1 Power-of-d choices

Consider N queues with independent service times distributed exponentially with mean 1, and a Poisson arrival to the system with rate $N\lambda$. Each arriving task selects a uniformly random subset of d out of N queues. The arriving task is dispatched to the smallest queue out of these selected d queues. We denote the queue length at the nth queue at time t as $X_n^N(t) : \Omega \to \mathcal{Z} \triangleq \mathbb{Z}_+$.

Definition 1.1. For the process X, the associated empirical distribution process is denoted by $A^N : \Omega \to \mathcal{M}_N(\mathbb{Z}_+)^{\mathbb{R}_+}$, and we define associated complementary distribution process for all $k \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$, as

$$B_k^N(t) \triangleq \sum_{j \geqslant k} A_j^N(t) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{X_n^N(t) \geqslant k\}}.$$

For a state $x \in \mathbb{Z}_+^N$, we denote the complementary distribution of states as $b \triangleq \sum_{k \in \mathbb{Z}_+} e_k \Big(\frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{x_n \geqslant k\}} \Big)$.

Proposition 1.2. *Defining $\mathcal{X} \triangleq \mathcal{Z}^N$, we observe that $X^N : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ is a continuous time Markov chain with the associated generator matrix defined for all $x, y \in \mathcal{X}$ as*

$$Q_{xy}^{X^N} = \begin{cases} \mathbb{1}_{\{x_n > 0\}}, & y = x - e_n, \\ \lambda \sum_{k \in \mathbb{Z}_+} \mathbb{1}_{\{x_n = k\}} \frac{\binom{Nb_k}{d} - \binom{Nb_{k+1}}{d}}{\binom{N}{d}(b_k - b_{k+1})} & y = x + e_n. \end{cases}$$

Proof. We observe that a transition takes place at one of the N queues when there is an arrival to the system with rate $N\lambda$, or a departure from one of the non-empty queues. It follows that the holding rate for the CTMC X in state x is $N\lambda + Nb_1(x)$. Let F be the random d subset of N queues chosen by an arrival. Then, an arrival to queue n takes place if $n \in F$ and $x_m > x_n$ for all queues $m \in F \setminus \{n\}$. Therefore, we can write the probability of an arrival to queue n as

$$\begin{aligned} &\frac{1}{\binom{N}{d}} \sum_{k \in \mathbb{Z}_+} \mathbb{1}_{\{x_n = k\}} \sum_{F \subseteq [N] \setminus \{n\} : |F| = d-1} \Big(\sum_{S \subseteq F} \frac{1}{|S|} \prod_{r \in S} \mathbb{1}_{\{x_r = k\}} \prod_{m \in F \setminus S} \mathbb{1}_{\{x_m \geqslant k+1\}} \Big) \\ &= \frac{1}{\binom{N}{d}} \sum_{k \in \mathbb{Z}_+} \mathbb{1}_{\{x_n = k\}} \sum_{j=0}^{d-1} \frac{1}{j+1} \binom{N(b_k - b_{k+1}) - 1}{j} \binom{Nb_{k+1}}{d-1-j} \\ &= \frac{1}{\binom{N}{d} N(b_k - b_{k+1})} \sum_{k \in \mathbb{Z}_+} \mathbb{1}_{\{x_n = k\}} \sum_{j=1}^{d} \binom{N(b_k - b_{k+1}) - 1}{j} \binom{Nb_{k+1}}{d-j}, \\ &= \frac{1}{\binom{N}{d} N(b_k - b_{k+1})} \sum_{k \in \mathbb{Z}_+} \mathbb{1}_{\{x_n = k\}} \Big[\binom{Nb_k}{d} - \binom{Nb_{k+1}}{d} \Big]. \end{aligned}$$

□

Corollary 1.3. *For $d = 2$, we can write the associated generator matrix for all $x, y \in \mathcal{X}$ as*

$$Q_{xy}^{X^N} = \begin{cases} \mathbb{1}_{\{x_n > 0\}}, & y = x - e_n, \\ \lambda \sum_{k \in \mathbb{Z}_+} \mathbb{1}_{\{x_n = k\}} \frac{b_k(Nb_k - 1) - b_{k+1}(Nb_{k+1} - 1)}{(N-1)(b_k - b_{k+1})}, & y = x + e_n. \end{cases}$$

Remark 1. We observe that $Q^{X^N}_{xy} = \sum_{n=1}^N f(x_n, y_n, a)\mathbb{1}_{\{y_n \neq x_n\}}$ and hence the empirical distribution evolves as a Markov process.

Proposition 1.4. *The complementary distribution process $B^N : \Omega \to [0,1]^{\mathbb{Z}_+}$ for power-of-d selection evolves as a Markov process with generator matrix defined for all $b, c \in [0,1]^{\mathbb{Z}_+}$*

$$Q^{B^N}_{b,c} \triangleq \begin{cases} N(b_k - b_{k+1}), & c = b - \frac{1}{N}e_k, \\ N\lambda\Big(\frac{\binom{Nb_{k-1}}{d} - \binom{Nb_k}{d}}{\binom{N}{d}}\Big), & c = b + \frac{1}{N}e_k. \end{cases}$$

Proof. We observe that only possible transitions are of the form $b \to c = b - \frac{1}{N}e_k$ and $b \to c = b + \frac{1}{N}e_k$ corresponding to a departure from a queue with length k and an arrival to a queue with length $k-1$ respectively. We can write

$$Q^{B^N}_{b,c} = \begin{cases} \sum_{n=1}^N \mathbb{1}_{\{x_n = k\}}, & c = b - \frac{1}{N}e_k, \\ \lambda \sum_{n=1}^N \mathbb{1}_{\{x_n = k-1\}} \frac{\binom{Nb_{k-1}}{d} - \binom{Nb_k}{d}}{\binom{N}{d}}, & c = b + \frac{1}{N}e_k. \end{cases}$$

□

2 Mean-field model

Definition 2.1. Let $\mathcal{X} \triangleq \mathcal{Z}^N$, and $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ an N interacting particle CTMC such that $A^N : \Omega \to \mathcal{M}_N(\mathcal{Z})^{\mathbb{R}_+}$ is Markov with generator matrix Q^{A^N}. The sequence of CTMCs $(A^N : N \in \mathbb{N})$ is called **density-dependent family** of CTMCs if the normalized generator matrix defined as $q \triangleq \frac{1}{N}Q^{A^N}$ is independent of N. For this family, we define the rate of change of distribution of states $f : \mathcal{M}(\mathcal{Z}) \to \mathbb{R}$ for each $a \in \mathcal{M}(\mathcal{Z})$ as

$$f(a) \triangleq \sum_{b \neq a} Q_{ab}(b - a). \tag{1}$$

Remark 2. Recall that $P_x(t) = \mathbb{E}_x[X_t]$ and $\frac{dP(t)}{dt} = QP(t)$. We observe that $f(a) \approx \mathbb{E}_a \frac{dA_t}{dt}$. To see this, we write for small t

$$\mathbb{E}_a[A_t - A_0] = \sum_{b \in \mathcal{M}(\mathcal{X})} P_{ab}(t)(b - a) \approx t \sum_{b \in \mathcal{M}(\mathcal{X})} Q_{ab}(b - a).$$

Definition 2.2. For a density dependent family of interacting particle CTMCs over particle state space $\mathcal{Z}$, with rate of change function $f : \mathcal{M}(\mathcal{Z}) \to \mathbb{R}$ defined in (1), the mean field model is defined as

$$\frac{d}{dt}x(t) = f(x(t)), \quad x(t) \in \mathcal{D} \subseteq \mathcal{M}(\mathcal{Z}). \tag{2}$$

Example 2.3 (SIS epidemic model). We can write the normalized generator matrix for SIS epidemic model as

$$q_{ab} = \begin{cases} a_0\alpha + a_0 a_1 \beta, & (b - a) = \frac{1}{N}(e_1 - e_0), \\ a_1, & (b - a) = \frac{1}{N}(e_0 - e_1). \end{cases}$$

In this case, we can write the rate change function $f : \mathcal{M}(\mathcal{Z}) \to \mathbb{R}$ as

$$f(a) = (a_0\alpha - a_1(1 - a_0\beta))(-e_0 + e_1).$$

Since $a_0 + a_1 = 1$, the mean field model can be written as

$$\frac{d}{dt}a_0(t) = -a_0\alpha - a_0(1 - a_0)\beta + (1 - a_0).$$

The stationary point of this ODE is the solution to the equation $a_0^2\beta - a_0(\alpha + \beta + 1) + 1 = 0$, which is

$$a_0^* = \frac{1 + \alpha + \beta}{2\beta} - \sqrt{\Big(\frac{1 + \alpha + \beta}{2\beta}\Big)^2 - \frac{1}{\beta}}.$$

Example 2.4 (Random multiple access). We can write the normalized generator matrix for random multiple access as

$$q_{ab} = \begin{cases} \langle c, a \rangle - c_z a_z, & (b-a) = \frac{1}{N}(e_{z+1} - e_z), \\ c_z a_z, & (b-a) = \frac{1}{N}(e_0 - e_z). \end{cases}$$

In this case, we can write the rate change function $f : \mathcal{M}(\mathcal{Z}) \to \mathbb{R}$ as

$$f(a) = \sum_{z=0}^{r-1} (\langle c, a \rangle - c_z a_z)(e_{z+1} - e_z) + \sum_{z=1}^{r-1} c_z a_z (e_0 - e_z) + \langle c, a \rangle (e_0 - e_r) = \sum_{z=0}^{r-1} c_z a_z (e_0 - e_{z+1}).$$

Since $a_0 + \sum_{z=1}^{r} a_z = 1$, the mean field model for $z \in [r]$ can be written as

$$\frac{d}{dt} a_z(t) = -c_{z-1} a_{z-1}(t).$$

Example 2.5 (Power-of-d choices). We can write the normalized generator matrix for random multiple access as

$$q_{bc} = \begin{cases} \lambda\Big(\frac{\binom{Nb_{k-1}}{d} - \binom{Nb_k}{d}}{\binom{N}{d}}\Big), & (c-b) = \frac{1}{N} e_k, \\ b_k - b_{k+1}, & (c-b) = -\frac{1}{N} e_k. \end{cases}$$

In this case, the rate of change of complementary distributions f, is given by

$$f(b) = \sum_{k \in \mathbb{N}} e_k (\lambda (b_{k-1}^d - b_k^d) - (b_k - b_{k+1})).$$

Recall that $b_0 = 1$, and the mean field model can be written for all $k \in \mathbb{N}$ as

$$\frac{d}{dt} b_k(t) = \lambda (b_{k-1}^d - b_k^d) - (b_k - b_{k+1}). \tag{3}$$

There is a unique stationary point of this ODE (3) with $\sum_{k \in \mathbb{Z}_+} b_k < \infty$ given by $b_k^* = \lambda^{\frac{d^k - 1}{d-1}}$ for all $k \in \mathbb{Z}_+$. We observe that $b_1^* = \lambda$ the mean load of the system.