Lecture-09: Kurtz's theorem: preliminaries

1 Preliminaries

Assumption 1.1. Consider a density-dependent family of CTMCs $((X^N : \Omega \to (\mathcal{Z}^N)^{\mathbb{R}}_+) : N \in \mathbb{N})$. For each N, state $x \in \mathcal{Z}^N$, empirical distribution of states $a(x) \in \mathcal{M}(\mathcal{Z})$, and $z, w \in \mathcal{Z}$, the transition rate $Q^{X^N_n}_{z,w} : \mathcal{M}(\mathcal{Z}) \to \mathbb{R}$ for a single particle n is Lipschitz continuous in the empirical distribution.

Proposition 1.2. *If Assumption 1.1 holds, then the following statements are true.*

1. *The maximum transition rate* $\bar{Q} \triangleq \max\left\{Q^{X^N_n}_{zw}(a) : z, w \in \mathcal{Z}, a \in \mathcal{M}(\mathcal{Z})\right\}$ *is finite.*
2. *The rate change function* $f : \mathcal{M}(\mathcal{Z}) \to \mathbb{R}^{\mathcal{Z}}$ *in Mckean-Vlasov ODE is Lipschitz continuous.*
3. *Mckean-Vlasov ODE is well-posed.*

Proof. From hypothesis, we have $\left|Q^{X^n}_{zw}(b) - Q^{X^N_n}_{zw}(a)\right| \leqslant M \left\|b-a\right\|_{\mathrm{TV}}$ for all $a, b \in \mathcal{M}_N(\mathcal{Z})$ and some finite $M \in \mathbb{R}_+$.

1. Fix $a, b \in \mathcal{M}_N(\mathcal{Z})$, then we get

$$\left|Q^{X^n}_{zw}(b)\right| \leqslant \left|Q^{X^N_n}_{zw}(a)\right| + \left|Q^{X^N_n}_{zw}(b) - Q^{X^N_n}_{zw}(a)\right| \leqslant \left|Q^{X^N_n}_{zw}(a)\right| + M\left\|b-a\right\|.$$

   The result follows from finiteness of $\mathcal{Z}$ and of the norm of difference $\|b-a\|$ for all $a, b \in \mathcal{M}(\mathcal{Z})$.

2. We can write the generator matrix for empirical distribution for transitioning states $b, c \in \mathcal{M}_N(\mathcal{Z})$ such that $N(c-b) = -e_z + e_w$, as

$$Q^{A^N}_{b,c} = \sum_{n=1}^{N}\sum_{z\in\mathcal{Z}} \mathbb{1}_{\{X^N_n = z\}} \sum_{w\neq z} \mathbb{1}_{\{N(c-b)=e_w-e_z\}} Q^{X^N_n}_{zw}(b) = \sum_{z\in\mathcal{Z}} N b_z \sum_{w\neq z} \mathbb{1}_{\{N(c-b)=e_w-e_z\}} Q^{X^N_1}_{zw}(b).$$

   Substituting this expression in the definition of $f(b) = \sum_{c\in\mathcal{M}_N(\mathcal{Z})} Q^{A^N}_{b,c}(c-b)$, we obtain

$$f(b) = \sum_{c\in\mathcal{M}_N(\mathcal{Z})}\sum_{z\in\mathcal{Z}} N b_z \sum_{w\neq z} \mathbb{1}_{\{N(c-b)=e_w-e_z\}} Q^{X^N_1}_{z,w}(b)\frac{1}{N}(e_w - e_z) = \sum_{z\in\mathcal{Z}} b_z \sum_{w\neq z} Q^{X^N_1}_{z,w}(b)(e_w - e_z).$$

   Thus, we can write the difference between rate functions evaluated at two different distributions as

$$f(b) - f(a) = \sum_{z\in\mathcal{Z}} b_z \sum_{w\neq z} Q^{X^N_1}_{z,w}(b)(e_w - e_z) - \sum_{z\in\mathcal{Z}} a_z \sum_{w\neq z} Q^{X^N_1}_{z,w}(a)(e_w - e_z).$$

   Since $Q^{X^N_1}_{z,w}(b) - Q^{X^N_1}_{z,w}(a) \leqslant M\left\|b-a\right\|_{\mathrm{TV}}$ and $\bar{Q} = \max_{z,w,a} Q^{X^N_1}_{z,w}(a)$, we get

$$\begin{aligned}
f(b) - f(a) &\leqslant M\left\|b-a\right\|_{\mathrm{TV}} \sum_{z\in\mathcal{Z}} b_z \sum_{w\neq z}(e_w + e_z) + \sum_{z\in\mathcal{Z}}(b_z - a_z)\sum_{w\neq z} Q^{X^N_1}_{z,w}(a)(e_w - e_z)\\
&\leqslant M\left\|b-a\right\|_{\mathrm{TV}} (\sum_{w\in\mathcal{Z}} e_w + (|\mathcal{Z}|-1)b) + \bar{Q}\sum_{z\in\mathcal{Z}} e_z\left|b_z - a_z\right|(|\mathcal{Z}|-1) + \bar{Q}\sum_{z\in\mathcal{Z}}\left|b_z - a_z\right|\sum_{w\in\mathcal{Z}} e_w\\
&\leqslant (M+\bar{Q})\left|\mathcal{Z}\right|\left\|b-a\right\|_{\mathrm{TV}}\sum_{z\in\mathcal{Z}} e_z.
\end{aligned}$$

3. Mckean-Vlasov ODE is well-posed.

□

Lemma 1.3. *Consider a homogeneous Poisson counting process* $\bar{N} : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ *with unit rate adapted to its natural filtration* $\mathcal{F}_\bullet$*, and define* $\bar{N} : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ *as* $\bar{N}_t \triangleq N_t - t$ *for all* $t \in \mathbb{R}_+$*. Then* $\bar{N}$ *is martingale adapted to* $\mathcal{F}_\bullet$*.*

Proof. We check the three conditions for $\bar{N}$ to be a martingale adapted to $\mathcal{F}_\bullet$.

1. We observe that $\mathbb{E}\,|N_t - t| \leqslant \mathbb{E}N_t + t < \infty$.
2. Since $\sigma(N_t) \subseteq \mathcal{F}_t$, it follows that $\sigma(\bar{N}_t) \subseteq \mathcal{F}_t$.
3. We can write $N_t - t = N_t - N_s + N_s - t$. Let $s < t$. From the increment independent property of N and the fact that $\sigma(N_s) \subseteq \mathcal{F}_s$, we observe $\mathbb{E}[N_t - t \mid \mathcal{F}_s] = \mathbb{E}[N_t - N_s] + N_s - t$. Further, since N is unit rate Poisson counting process, it follows that $\mathbb{E}[N_t - N_s] = t - s$. Therefore, we have $\mathbb{E}[N_t - t \mid \mathcal{F}_s] = N_s - s$ for all $s < t$.

□

Lemma 1.4. *Consider a homogeneous Poisson counting process* $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ *with unit rate adapted to its natural filtration* $\mathcal{F}_\bullet$ *and* $\theta > 0$*. We define* $Y, Z : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ *as* $Y_t \triangleq e^{\theta(N_t - t)} = 1/Z_t$ *for all* $t \in \mathbb{R}_+$*. Then* Y, Z *are submartingales adapted to* $\mathcal{F}_\bullet$*.*

Proof. From Lemma 1.3, we know that centered Poisson counting process $\bar{N}$ is a martingale adapted to $\mathcal{F}_\bullet$. Further, we observe that functions $x \mapsto e^{\theta x}$ and $x \mapsto e^{-\theta x}$ are convex for $\theta > 0$. In addition, we have

$$\mathbb{E}\left|e^{\theta(N_t - t)}\right| = \mathbb{E}e^{\theta(N_t - t)} = e^{t(e^\theta - 1 - \theta)} < \infty, \qquad \mathbb{E}\left|e^{-\theta(N_t - t)}\right| = \mathbb{E}e^{-\theta(N_t - t)} = e^{t(e^{-\theta} - 1 + \theta)} < \infty.$$

It follows that Y, Z are submartingales adapted to $\mathcal{F}_\bullet$. □

Lemma 1.5. *Consider a submartingale* $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ *and a stopping time* $\tau : \Omega \to \mathbb{N}$ *adapted to filtration* $\mathcal{F}_\bullet$*. If* $\tau \leqslant n$ *almost surely, then* $\mathbb{E}X_1 \leqslant \mathbb{E}X_\tau \leqslant \mathbb{E}X_n$*.*

Proof. From the fact that X is a submartingale and τ is a stopping time, both adapted to $\mathcal{F}_\bullet$, we get

$$\mathbb{E}[X_n \mathbb{1}_{\{\tau = k\}} \mid \mathcal{F}_k] = \mathbb{1}_{\{\tau = k\}}\mathbb{E}[X_n \mid \mathcal{F}_k] \geqslant X_\tau \mathbb{1}_{\{\tau = k\}}.$$

Summing over $k \in [n]$, then taking expectation on both sides, it follows from the linearity and monotonicity preserving property of expectation that $\mathbb{E}X_n \sum_{k=1}^n \mathbb{1}_{\{\tau = k\}} \geqslant \mathbb{E}\sum_{k=1}^n X_\tau \mathbb{1}_{\{\tau = k\}}$. Since $\tau \leqslant n$ almost surely, we have $X_\tau = \sum_{k=1}^n X_\tau \mathbb{1}_{\{\tau = k\}}$ and $X_n = X_n \sum_{k=1}^n \mathbb{1}_{\{\tau = k\}}$ almost surely. □

Lemma 1.6 (Doob). *For a submartingale* $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *adapted to a filtration* $\mathcal{F}_\bullet$*,* $P\left\{\max_{i \in [n]} X_i > x\right\} \leqslant \frac{\mathbb{E}X_n}{x}$ *for all* $x > 0$*.*

Proof. We define a random time $\tau_x \triangleq \inf\{i \in \mathbb{N} : X_i > x\}$ and $\tau \triangleq \tau_x \wedge n$. It follows that τ is a stopping time adapted to $\mathcal{F}_\bullet$ and $\tau < n$ almost surely. We observe that $\left\{\max_{i \in [n]} X_i > x\right\} = \cup_{i \in [n]}\{X_i > x\} = \{X_\tau > x\}$. From Markov inequality for non-negative random variables, we have $P\{X_\tau > x\} \leqslant \frac{\mathbb{E}X_\tau}{x}$. The result follows from Lemma 1.5. □

Lemma 1.7 (Doob). *For a submartingale* $X : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ *adapted to a filtration* $\mathcal{F}_\bullet$*,* $P\left\{\sup_{t \in [0,T]} X_t > x\right\} \leqslant \frac{\mathbb{E}X_T}{x}$ *for all* $x > 0$*.*

Corollary 1.8. *For a submartingale* $X : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ *adapted to a filtration* $\mathcal{F}_\bullet$*,* $P\left\{\sup_{t \in [0,T]} X_t > x\right\} \leqslant \frac{\mathbb{E}(X_T \vee 0)}{x}$ *for all* $x > 0$*.*

Proof. The function $f : \mathbb{R} \to \mathbb{R}_+$ defined as $f(x) \triangleq x \vee 0$ is non-negative and convex and hence $Y : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ defined as $Y_t \triangleq f(X_t)$ for all $t \in \mathbb{R}_+$, is a sub-martingale adapted to $\mathcal{F}_\bullet$. The result follows from Lemma 1.7 and the fact that $\left\{\sup_{t \in [0,T]} X_t > x\right\} = \left\{\sup_{t \in [0,T]} Y_t > x\right\}$ for all $x > 0$. □

Definition 1.9. We define $h : [-1, \infty) \to \mathbb{R}_+$ as $h(t) \triangleq (1 + t)\ln(1 + t) - t$ for all $t \geqslant -1$.

Remark 1. The function h defined in Definition 1.9 is positive and increasing for $t > 0$ and $\lim_{t\to 0} h(t) = 0$. Further, we observe that $h(-t) \geqslant h(t)$ for $t \in [0,1]$.

Lemma 1.10. *Consider a homogeneous Poisson counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with unit rate. Then for any $\epsilon, T > 0$, we have $P\left\{\sup_{t\in[0,T]} |N_t - t| > \epsilon\right\} \leqslant 2e^{-Th(\frac{\epsilon}{T})}$.*

Proof. From the union bound applied to event $\left\{\sup_{t\in[0,T]} |N_t - t| > \epsilon\right\}$, we obtain

$$P\left\{\sup_{t\in[0,T]} |N_t - t| > \epsilon\right\} \leqslant P\left\{\sup_{t\in[0,T]} e^{\theta(N_t-t)} > e^{\theta\epsilon}\right\} + P\left\{\sup_{t\in[0,T]} e^{-\theta(N_t-t)} > e^{\theta\epsilon}\right\}.$$

For the Poisson counting process N, consider the submartingales Y, Z defined in Lemma 1.4 adapted to the natural filtration $\mathcal{F}_\bullet$ of N. From Doob's inequality of Lemma 1.7, we observe that

$$P\left\{\sup_{t\in[0,T]} Y_t > e^{\theta\epsilon}\right\} \leqslant e^{-\theta\epsilon}\mathbb{E}Y_T, \qquad P\left\{\sup_{t\in[0,T]} Z_t > e^{\theta\epsilon}\right\} \leqslant e^{-\theta\epsilon}\mathbb{E}Z_T.$$

From the moment generating function for a Poisson counting process with unit rate, we get

$$\mathbb{E}Y_T = \mathbb{E}e^{\theta(N_T-T)} = e^{T(e^\theta-1-\theta)}, \qquad \mathbb{E}Z_T = \mathbb{E}e^{-\theta(N_T-T)} = e^{T(e^{-\theta}-1+\theta)}.$$

We observe that $e^{-\theta\epsilon}\mathbb{E}Y_T = \exp(-\theta\epsilon + T(e^\theta - 1 - \theta))$ is minimized at $\theta^* = \ln(\frac{\epsilon}{T} + 1) > 0$. For this value of θ^*, we have

$$e^{-\theta^*\epsilon}\mathbb{E}Y_T(\theta^*) = \exp(-\epsilon\ln(\frac{\epsilon}{T}+1) + T(\frac{\epsilon}{T} - \ln(\frac{\epsilon}{T}+1))) = e^{-Th(\frac{\epsilon}{T})}.$$

Similarly, $e^{-\theta\epsilon}\mathbb{E}Z_T = \exp(-\theta\epsilon + T(e^{-\theta} - 1 + \theta))$ is minimized at $\theta^{**} = -\ln(1 - \frac{\epsilon}{T}) > 0$. For this value of θ^{**}, we have

$$e^{-\theta^{**}\epsilon}\mathbb{E}Y_T(\theta^{**}) = \exp(\epsilon\ln(1-\frac{\epsilon}{T}) + T(-\frac{\epsilon}{T} - \ln(1-\frac{\epsilon}{T}))) = e^{-Th(-\frac{\epsilon}{T})}.$$

Since $h(-\frac{\epsilon}{T}) \geqslant h(\frac{\epsilon}{T})$ for $\epsilon \leqslant T$, the result follows. □

Definition 1.11. For a rate function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, we can define $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ as $\Lambda_t \triangleq \int_0^s \lambda(s)ds$ for all $t \in \mathbb{R}_+$. We can define the extension $\Lambda : \mathcal{B}(\mathbb{R}_+) \to \mathbb{R}_+$ as $\Lambda(B) = \int_{s\in B} \lambda(s)ds$ for all Borel sets $B \in \mathcal{B}(\mathbb{R}_+)$.

Lemma 1.12 (Time change). *Let $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ be a homogeneous Poisson counting process with unit rate, and $\Lambda : \mathcal{B}(\mathbb{R}_+) \to \mathbb{R}_+$ be a measure with rate $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$. The counting process $M : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ defined as $M_t \triangleq N_{\Lambda_t}$ for all $t \in \mathbb{R}_+$ is non-homogeneous Poisson with instantaneous rate λ.*

Proof. We denote the natural filtration for N and M as $\mathcal{F}_\bullet$ and $\mathcal{G}_\bullet$ respectively. From the independent increment property of N, we obtain

$$\mathbb{E}[M_t - M_s \mid \mathcal{G}_s] = \mathbb{E}[N_{\Lambda_t} - N_{\Lambda_s} \mid \mathcal{F}_{\Lambda_s}] = \mathbb{E}[N_{\Lambda_t} - N_{\Lambda_s}] = \mathbb{E}[M_t - M_s].$$

It follows that M has independent increment property. Further with mean number of points in $(s,t]$ under M is given by measure $\Lambda(s,t]$. □