Lecture-04: Invariant Distribution of Markov Processes

1 Class Properties

Definition 1.1. For a CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ defined on the countable state space $\mathcal{X} \subseteq \mathbb{R}$, we say a state y is **reachable** from state x if $P_{xy}(t) > 0$ for some $t > 0$, and we denote $x \to y$. If two states $x, y \in \mathcal{X}$ are reachable from each other, we say that they **communicate** and denote it by $x \leftrightarrow y$.

Lemma 1.2. *Communication is an equivalence relation.*

Definition 1.3. Communication equivalence relation partitions the state space $\mathcal{X}$ into equivalence classes called **communicating classes**. A CTMC with a single communicating class is called **irreducible**.

Theorem 1.4. *A regular CTMC and its embedded DTMC have the same communicating classes.*

Proof. It suffices to show that $x \to y$ for the regular Markov process iff $x \to y$ in the embedded chain. If $x \to y$ for the embedded chain, then there exists a path $x = x_0, x_1, \dots, x_n = y$ such that $p_{x_0x_1}p_{x_1x_2}\dots p_{x_{n-1}x_n} > 0$ and $0 < \nu_{x_0}\nu_{x_1}\dots\nu_{x_{n-1}}$. It follows that S_n is a stopping time and sum of n independent exponential random variables with rates $\nu_{x_0}, \dots, \nu_{x_{n-1}}$, and we can write

$$P_{xy}(t) \geqslant P\left\{X_0 = x_0, X_{S_1} = x_1, \dots, X_{S_n} = x_n, N_t = n\right\} = \prod_{k=0}^{n-1} p_{x_kx_{k+1}} \mathbb{E}[\mathbb{1}_{\{N_t=n\}} \mid \cap_{i=0}^{n}\left\{Z_i = x_i\right\}] > 0.$$

Conversely, if the states y is not reachable from state x in embedded chain, then it won't be reachable in the regular CTMC. ☐

Corollary 1.5. *A regular CTMC is irreducible iff its embedded DTMC is irreducible.*

Remark 1. There is no notion of periodicity in CTMCs since there is no fundamental time-step that can be used as a reference to define such a notion. In fact, for any state $x \in \mathcal{X}$ of a non-instantaneous homogeneous CTMC we have $P_{xx}(t) > e^{-\nu_x t} > 0$ for all $t \geqslant 0$.

1.1 Recurrence and transience

Consider a continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ and its embedded discrete time Marko chain $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$.

Definition 1.6. Let $k \in \mathbb{N}$. For any state $x \in \mathcal{X}$, we denote the kth return time to state x by $\tau_x^+(k)$ and kth sojourn time in state x by $Y_k^{(x)}$. We inductively define $\tau_x^+(0) \triangleq 0$ and

$$\tau_x^+(k) \triangleq \inf\left\{t > \tau_x^+(k-1) + Y_k^{(x)} : X_t = x\right\}.$$

Definition 1.7. A state $x \in \mathcal{X}$ is said to be **recurrent** if $P_x\left\{\tau_x^+(1) < \infty\right\} = 1$ and **transient** if $P_x\left\{\tau_x^+(1) < \infty\right\} < 1$. Furthermore, a recurrent state x is said to be **positive recurrent** if $\mathbb{E}_x\tau_x^+(1) < \infty$ and **null recurrent** if $\mathbb{E}_x\tau_x^+(1) = \infty$.

Definition 1.8. We denote the number of visits to state y during kth successive visit to state x by

$$N_{xy}(k) \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{[\tau_x^+(k-1), \tau_x^+(k))}(S_n)\mathbb{1}_{\{Z_n=y\}}.$$

The total number of visits to all states during kth successive visit to state x is defined as

$$N_x(k) \triangleq \sum_{y\in\mathcal{X}} N_{xy}(k) = \sum_{n\in\mathbb{N}} \mathbb{1}_{[\tau_x^+(k-1), \tau_x^+(k))}(S_n).$$

The total number to visits to all states before kth return to state x is defined as $S_x^+(k) \triangleq \sum_{j=1}^{k} N_x(j)$.

Lemma 1.9. *We define the jth sojourn time in state y during kth return duration $[\tau_x^+(k-1), \tau_x^+(k))$ for state x as $Y_{kj}^{(y)}$. Then, the k return time to state x is*

$$\tau_x^+(k) = \tau_x^+(k-1) + \sum_{y \in \mathcal{X}} \sum_{j=1}^{N_{xy}(k)} Y_{kj}^{(y)}.$$

Proof. Since $1 = \mathbb{1}_{\{X_t \in \mathcal{X}\}} = \mathbb{1}_{\cup_{y \in \mathcal{X}}\{X_t = y\}} = \sum_{y \in \mathcal{X}} \mathbb{1}_{\{X_t = y\}}$, we can write the following equality

$$\tau_x^+(k) = \tau_x^+(k-1) + \int_{\tau_x^+(k-1)}^{\tau_x^+(k)} \sum_{y \in \mathcal{X}} \mathbb{1}_{\{X_t = y\}} dt.$$

Further, we can write $\mathbb{1}_{\{X_t = y\}} = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{Z_n = y\}} \mathbb{1}_{[S_n, S_{n+1})}(t)$. Interchanging sum and integral using monotone convergence theorem, we obtain

$$\tau_x^+(k) = \tau_x^+(k-1) + \sum_{y \in \mathcal{X}} \sum_{n \in \mathbb{N}} \mathbb{1}_{\{Z_n = y\}} (S_{n+1} - S_n) \mathbb{1}_{\left\{S_x^+(k-1) \leqslant n < S_x^+(k)\right\}}.$$

We observe that $V_{xy}(k) \triangleq \{n \in \mathbb{N} : S_x^+(k-1) \leqslant n < S_x^+(k), Z_n = y\}$ is the set of transitions which correspond to visits to state y during kth return time to state x, and $N_{xy}(k) = |V_{xy}(k)|$. Further, the duration $S_{n+1} - S_n$ is the sojourn time in state Z_n. Therefore, the result follows. □

Theorem 1.10. *An irreducible pure jump CTMC is recurrent iff its embedded DTMC is recurrent.*

Proof. A regular CTMC is pure jump by definition. Further, a regular CTMC is irreducible iff embedded DTMC is irreducible from Corollary 1.5. There is nothing to prove for $|\mathcal{X}| = 1$. Hence, we assume $|\mathcal{X}| \geqslant 2$ without loss of generality.

Suppose that the embedded Markov chain $Z : \Omega \to \mathcal{X}^{\mathbb{N}}$ is recurrent. Since the embedded chain is irreducible and recurrent, CTMC has no absorbing states. This implies $N_{xy}(1)$ and $N_x(1)$ are finite almost surely, and the random sequence $Y^{(y)} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* exponential with rate $\nu_y \in (0, \infty)$, and sequences $Y^{(y)}$ are independent for each state $y \in \mathcal{X}$. Since the recurrence time $\tau_x^+(1)$ is an a.s. finite sum of finite random variables, it follows that $\tau_x^+(1)$ is finite almost surely.

Conversely, if the embedded Markov chain is not recurrent, it has a transient state $x \in \mathcal{X}$ for which $P_x\{N_x = \infty\} > 0$. By the same argument, $P_x\{\tau_x^+ = \infty\} > 0$ and hence the CTMC is not recurrent. □

Corollary 1.11. *Recurrence is a class property.*

Theorem 1.12. *Consider an irreducible positive recurrent discrete time Markov chain $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with transition probability matrix $p \in \mathcal{M}(\mathcal{X})^{\mathcal{X}}$ and invariant distribution $u \in \mathcal{M}(\mathcal{X})$. Then,*

$$u_y = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{Z_n = y\}} = \frac{\mathbb{E}_x N_{xy}(k)}{\mathbb{E}_x N_x(k)} = u_x \mathbb{E}_x N_{xy}(k).$$

Proof. Let $Z_0 = x$. For a homogeneous Markov chain Z, the random sequence $S_x^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ is a renewal sequence, and the number of visits $N_x(k)$ to all states before the kth return to state x is the kth inter-return time to state x. The number of visits to state y between two successive visits to state x is

$$N_{xy}(k) = \sum_{n=S_x^+(k-1)+1}^{S_x^+(k)} \mathbb{1}_{\{Z_n = y\}}.$$

We can consider $N_{xy}(k)$ as the reward in the kth renewal duration. The result follows from the renewal reward theorem and the fact that $N_{xx}(k) = 1$ for all $k \in \mathbb{N}$ and $x \in \mathcal{X}$. □

Theorem 1.13. *Consider an irreducible recurrent continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with sojourn time rates $\nu \in \mathbb{R}_+^{\mathcal{X}}$ and transition matrix $p \in \mathcal{M}(\mathcal{X})^{\mathcal{X}}$ for the embedded Markov chain $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$. Let $u \in \mathbb{R}_+^{\mathcal{X}}$ be any strictly positive solution of $u = up$, then for each state $x \in \mathcal{X}$*

$$\mathbb{E}_x \tau_x^+(1) = \frac{1}{u_x} \sum_{y \in \mathcal{X}} \frac{u_y}{\nu_y}. \tag{1}$$

Further, the process X is positive recurrent iff $\sum_{x \in \mathcal{X}} \frac{u_x}{\nu_x} < \infty$.

Proof. Let $X_0 = x \in \mathcal{X}$. Recall that $Y_k^{(x)}$ denotes the kth sojourn time of the Markov process X in state x, and the random sequence $Y^{(x)} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* with common exponential distribution of rate ν_x. From Lemma 1.9, the first visit time to state x in terms of $N_{xy}(1)$ and sojourn times $Y_k^{(y)}$ for each state $y \in \mathcal{X}$, is $\tau_x^+(1) = \sum_{y\in\mathcal{X}} \sum_{k=1}^{N_{xy}(1)} Y_k^{(y)}$. We recall that jump chain Z and sojourn times are independent given the initial state, and hence $N_{xy}(1)$ and $Y^{(y)}$ sequences are independent for each state $y \neq x$. From taking expectations on both sides, exchanging summation and expectations by the application of monotone convergence theorem for positive random variables, we get $\mathbb{E}_x\tau_x^+(1) = \sum_{y\in\mathcal{X}} \mathbb{E}Y_k^{(y)}\mathbb{E}_x N_{xy}$. To show (1), it suffices to show that $\mathbb{E}_x N_{xy}(k) = \frac{u_y}{u_x}$.

The embedded Markov chain Z inherits the irreducibility and recurrence of the Markov process X from Corollary 1.5 and Theorem 1.10. For irreducible and recurrent Markov chain Z with transition matrix p and any strictly positive solution to $u = up$, we have $\mathbb{E}_x N_{xy}(k) = \frac{u_y}{u_x}$ from Theorem 1.12.

Since u is strictly positive, it follows that $\mathbb{E}_x\tau_x^+(1) < \infty$ iff $\sum_{y\in\mathcal{X}} \frac{u_y}{\nu_y} < \infty$. □

Remark 2. For an irreducible regular CTMC X, the embedded Markov chain Z is irreducible and recurrent. If Z with transition matrix p is positive recurrent, then there exists a strictly positive solution equilibrium distribution $u \in \mathcal{M}(\mathcal{X})$ such that $u = up$. However, it is possible that rates $\nu \in \mathbb{R}_+^{\mathcal{X}}$ ensure that $\sum_{y\in\mathcal{X}} \frac{u_y}{\nu_y} = \infty$, in which case X is null recurrent.

2 Invariant Distribution

Remark 3. For a homogeneous Markov process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with probability transition kernel $P : \mathbb{R}_+ \to \mathcal{M}(\mathcal{X})^{\mathcal{X}}$, we denote the marginal distribution of random variable X_t at time t by $\nu(t) \in \mathcal{M}(\mathcal{X})$, where for each time $t \in \mathbb{R}_+$

$$\nu(t) = \nu(0)P(t).$$

In general, we can write $\nu(s + t) = \nu(s)P(t)$. Hence, if there exists a stationary distribution $\pi \triangleq \lim_{s\to\infty} \nu(s)$ for this process X, then we would have $\pi = \pi P(t)$ for all times $t \in \mathbb{R}_+$.

Definition 2.1. A distribution $\pi \in \mathcal{M}(\mathcal{X})$ is an **invariant distribution** of a homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with probability transition kernel $P : \mathbb{R}_+ \to \mathcal{M}(\mathcal{X})^{\mathcal{X}}$ if $\pi P(t) = \pi$ for all $t \in \mathbb{R}_+$.

Corollary 2.2. *For a homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with generator matrix Q, a distribution $\pi \in \mathcal{M}(\mathcal{X})$ is an equilibrium distribution iff $\pi Q = 0$.*

Proof. Recall that we can write the transition probability matrix $P(t)$ at any time $t \in \mathbb{R}_+$ in terms of generator matrix Q as $P(t) = e^{tQ}$. Using the exponentiation of a matrix, we can write

$$\pi P(t) = \pi e^{tQ} = \pi + \sum_{n\in\mathbb{N}} \frac{t^n}{n!}\pi Q^n, \quad t \in \mathbb{R}_+.$$

Therefore, $\pi Q = 0$ iff π is an equilibrium distribution of the Markov process X. □

Theorem 2.3. *Let $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ be an irreducible recurrent homogeneous CTMC with probability transition kernel $P : \mathbb{R}_+ \to \mathcal{M}(\mathcal{X})^{\mathcal{X}}$, the transition rate sequence $\nu \in \mathbb{R}_+^{\mathcal{X}}$, and the transition matrix for embedded jump chain $p \in \mathcal{M}(\mathcal{X})$. Then for all states $x, y \in \mathcal{X}$ the $\lim_{t\to\infty} P_{xy}(t)$ exists, this limit is independent of the initial state $x \in \mathcal{X}$ and denoted by π_y. Let u be any strictly positive invariant measure such that $u = up$. If $\sum_{x\in\mathcal{X}} \frac{u_x}{\nu_x} = \infty$, then $\pi_x = 0$ for all $x \in \mathcal{X}$. If $\sum_{x\in\mathcal{X}} \frac{u_x}{\nu_x} < \infty$ then for all $y \in \mathcal{X}$,*

$$\pi_y = \frac{\frac{u_y}{\nu_y}}{\sum_{x\in\mathcal{X}} \frac{u_x}{\nu_x}} = \frac{\nu_y^{-1}}{\mathbb{E}_y\tau_y^+}.$$

Proof. Fix a state $y \in \mathcal{X}$, and define a process $W : \Omega \to \{0, 1\}^{\mathbb{R}_+}$ such that $W_t = \mathbb{1}_{\{X_t=y\}}$. Then, from the regenerative property of the homogeneous CTMC and renewal reward theorem, we have

$$\lim_{t\to\infty} P_x \{X_t = y\} = \frac{\mathbb{E}Y_k^{(y)}}{\mathbb{E}_y\tau_y^+(k)} = \frac{\nu_y^{-1}}{\mathbb{E}_y\tau_y^+}.$$

□