Lecture-01: Sets

1 Introduction

Why do we need probability theory?

1. Good abstraction for phenomena that are too complex to fully model, e.g. epidemic spread in populations, motion of molecules in gas (statistical physics), prices of financial instruments such as stocks, options, commodities etc., evolution of photon states in Quantum mechanics. In Quantum mechanics, randomness is an intrinsic feature of the model.
2. Deliberate uses of probability can justify goodness of certain algorithms, and their statistical limitations, e.g. Training AI models using stochastic (random sampling) gradient descent, statistical sampling for censuses, probability in this case

2 Mathematical preliminaries

We will study basic set theory and functions, since events are sets and probability is a function from a collection of events to $[0,1]$.

2.1 Sets

A *set* is a collection of well-defined objects called *elements*. There are no *duplicate* elements in a set. A set can be specified explicitly or implicitly.

Definition 2.1 (Membership). We write $x \in S$, if set S contains the element x.

Definition 2.2. We denote the set of first N positive integers by $[N] \triangleq \{1,2,\dots,N\}$, the set of positive integers (natural numbers) by $\mathbb{N} \triangleq \{1,2,\dots\}$, the set of integers by $\mathbb{Z} \triangleq \{\dots,-1,0,1,\dots\}$, the set of non-negative integers by $\mathbb{Z}_+ \triangleq \{0,1,\dots\}$, the set of rational numbers by $\mathbb{Q}$, the set of reals by $\mathbb{R}$, and the set of non-negative reals by $\mathbb{R}_+$.

Example 2.3. An example of an explicitly specified set is $S \triangleq \{1,2,6,5\}$.

Example 2.4. An example of an implicitly defined set is $U \triangleq \{n \in \mathbb{N} : n < 5 \text{ or } n > 10\}$, which is the set of all natural numbers such that any element of this set is either less than 5 or greater than 10.

Example 2.5. An example of an implicitly defined set is $T \triangleq \{x \in \mathbb{R} : x \leqslant 5, x > 2\}$, which is the set of all real numbers such that the element is less than and equal to 5 or greater than 2.

Definition 2.6 (Subsets). Set A is a subset of set B denoted as $A \subseteq B$, if each element of A is an element of B. Set A is a proper subset of B denoted as $A \subset B$ if (a) $A \subseteq B$ and (b) there is an element of B that doesn't belong to A. Sets A and B are identical if $A \subseteq B$ and $B \subseteq A$, and we denote $A = B$.

Remark 1. Empty set $\emptyset$ is a subset of all sets.

Definition 2.7. Power set of a set A is the collection of all subsets of A and denoted by $\mathcal{P}(A) \triangleq \{B : B \subseteq A\}$.

Example 2.8. Consider a set $\Omega = \{1,2,3\}$, then $\mathcal{P}(\Omega) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \Omega\}$.

Definition 2.9 (Product sets). Cartesian products of sets A and B is the set of all tuples (x,y) where $x \in A$ and $y \in B$, and denoted by $A \times B \triangleq \{(x,y) : x \in A, y \in B\}$.

Example 2.10. The cartesian product of two finite sets $A \triangleq [3]$ and $B \triangleq \{a,b\}$ is given by $A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$.

Example 2.11. The cartesian product of sets $A \triangleq [2]$ and $B \triangleq \emptyset$ is given by $A \times B = \emptyset$.

Definition 2.12 (Union). The *union* of two sets A, B is defined as $A \cup B \triangleq \{x : x \in A \text{ or } x \in B\}$. In general, let $(A_i : i \in I)$ be a *collection of sets* indexed by an index set I, then the union of this family of sets is defined

$$\cup_{i \in I} A_i \triangleq \left\{x : \text{ there exists a } j \in I \text{ such that } x \in A_j\right\}.$$

Example 2.13. The union of a family of sets $(A_i : i \in I)$ indexed by a finite index set $I \triangleq [n]$ is given by $\cup_{i \in I} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \left\{x : \text{ there exists a } j \in [n] \text{ such that } x \in A_j\right\}$..

Example 2.14. Consider index set $I \triangleq \mathbb{N}$ and family of sets $(A_i : i \in I)$.

1. Let $A_i = \{i\}$ for each $i \in \mathbb{N}$, then $\cup_{i \in I} A_i = \mathbb{N}$.
2. Let $A_i \triangleq [i, i+1]$ for each $i \in I$. Then, $\cup_{i \in I} A_i = [1, \infty)$.

Definition 2.15 (Intersection). The *intersection* of two sets A, B is defined as $A \cap B \triangleq \{x : x \in A \text{ and } x \in B\}$. In general, let $(A_i : i \in I)$ be a *collection of sets* indexed by an index set I, then the intersection of this family of sets is defined as

$$\cap_{i \in I} A_i \triangleq \left\{x : x \in A_j \text{ for each } j \in I\right\}.$$

Example 2.16. Consider index set $I \triangleq \mathbb{N}$ and family of sets $(A_i : i \in I)$ such that $A_i \triangleq [0, \frac{1}{i}]$ for each $i \in I$. Then, $\cap_{i \in I} A_i = \{0\}$. To see this, we first observe that $0 \in A_i$ for each $i \in I$, and hence $\{0\} \subseteq \cap_{i \in I} A_i$. To show the converse, we will show that $\cap_{i \in I} A_i \subseteq [0,1] \setminus (0,1]$. We observe that $A_i \subseteq [0,1]$ for each $i \in \mathbb{N}$, and hence it suffices to show that $\cap_{i \in I} A_i \cap (0,1] = \emptyset$. Consider an arbitrary $x \in (0,1]$, then $\frac{1}{x}$ is finite and there exists $i \in \mathbb{N}$ such that $i > \frac{1}{x}$. It follows that $x > \frac{1}{i}$ and hence $x \notin A_j$ for all $j \geqslant i$. In partoicular, $x \notin \cap_{i \in I} A_i$. Since the choice of x was arbitrary the result follows.

Definition 2.17 (Complement). Consider a set Ω considered the universal set and a subset $S \subseteq \Omega$. The *complement* of the set $S \subseteq \Omega$ is defined as $S^c \triangleq \{x \in \Omega : x \notin S\}$.

Definition 2.18 (Set difference). Consider two sets $A, B \subseteq \Omega$, then the *set difference* $A \setminus B$ or the *relative complement* of B in universal set A is defined as $A \setminus B \triangleq A \cap B^c \triangleq \{x \in A : x \notin B\}$.

Example 2.19. Consider two sets $A = [1,3]$ and $B = [2,4]$, both subsets of $\mathbb{R}$. Then the set difference $A \setminus B = A \cap B^c = [1,3] \cap ((-\infty, 2) \cup (4, \infty) = [1,2) = \{x \in \mathbb{R} : 1 \leqslant x < 2\}$.

Example 2.20. If $A \cap B = \emptyset$, then $A \setminus B$ = A.