Lecture-02: Sample and Event Space

1 Mathematical preliminaries

1.1 Properties of set operations

Proposition 1.1. *Let I be an arbitrary index set and $(A_i \subseteq \Omega : i \in I)$ a family of sets. Let $B \subseteq \Omega$ be another set.*
(a) $\Big(\cap_{i\in I} A_i\Big) \cup B = \cap_{i\in I}(A_i \cup B)$.
(b) $\Big(\cap_{i\in I} A_i\Big) \cap B = \cap_{i\in I}(A_i \cap B)$.
(c) ***De Morgan's Law:*** $\Big(\cap_{i\in I} A_i\Big)^c = \cup_{i\in I} A_i^c, \Big(\cup_{i\in I} A_i\Big)^c = \cap_{i\in I} A_i^c$.

Proof. Recall that to show a set $S_1 = S_2$, we have to show that $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$. To show that $S_1 \subseteq S_2$, we take any element $x \in S_1$ and show that $x \in S_2$. Similarly, we show the converse.
(a) Let $x \in \Big(\cap_{i\in I} A_i\Big) \cup B$, then either (i) $x \in B$ or (ii) $x \in A_i$ for each $i \in I$. If $x \in B$, then $x \in A_i \cup B$ for each $i \in I$ and hence $x \in \cap_{i\in I}(A_i \cup B)$. If $x \in A_i$ for each $i \in I$, then $x \in A_i \cup B$ for each $i \in I$ and hence $x \in \cap_{i\in I}(A_i \cup B)$.
Similarly, we can show the converse.

□

1.2 Functions and cardinality

Definition 1.2 (Function). For sets A, B, a function f from set B to set A is a subset of cartesian product $B \times A$ such that for each element $b \in B$ there is a unique element $a \in A$ such that $(b,a) \in f$. This unique element is denoted by $a \triangleq f(b)$. That is, $\{(b, f(b)) : b \in B\} \subseteq B \times A$. The set B and A are called the **domain** and **co-domain** of function f, and the set $f(B) = \{f(b) \in A : b \in B\} \subseteq A$ is called the **range** of function f. The collection of all A-valued functions with the domain B is denoted by A^B.

Example 1.3. Consider set $B = \{1,2,3\}$ and set $A = \{a,b\}$.
(a) The cartesian product of two sets is $B \times A = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$.
(b) The following subset of the cartesian product $\{(1,a),(2,a),(3,b)\} \subseteq B \times A$ corresponds to a function $f : B \to A$ such that $f(1) = f(2) = a, f(3) = b$. We will denote this function by ordered tuple (aab).
(c) The following subset $\{(1,a),(2,b),(2,c)\} \subseteq B \times A$ doesn't correspond to a function.
(d) The collection A^B is defined by the set of ordered tuples

$$\{(aaa),(aab),(aba),(abb),(baa),(bab),(bba),(bbb)\}.$$

Definition 1.4 (Inverse Map). For a function $f \in A^B$, we define **set inverse map** $f^{-1} \in \mathcal{P}(B)^{\mathcal{P}(A)}$ for each $C \in \mathcal{P}(A)$ as $f^{-1}(C) \triangleq \{b \in B : f(b) \in C\}$.

Example 1.5. Let $B = \{1,2,3\}$ and $A = \{a,b\}$ and f be denoted by the ordered tuple (aba), then $f^{-1}(\{a\}) = \{1,3\}$ and $f^{-1}(\{b\}) = \{2\}$.

Definition 1.6 (Injective, surjective, bijective). A function $f \in A^B$ is

injective: if for any distinct $b \neq c \in B$, we have $f(b) \neq f(c)$,

surjective: if $f(B) = A$, and

bijective: if it is injective and surjective.

Example 1.7. **injective:** Let $B = \{1,2,3\}$ and $A = \{a,b,c,d\}$. Then (abc) is an injective function.

surjective: Let $B = \{1,2,3,4\}$ and $A = \{a,b,c\}$. Then $(abca)$ is a surjective function.

bijective: Let $B = \{1,2,3\}$ and $A = \{a,b,c\}$. Then (abc) is a bijective function.

Definition 1.8 (Cardinality). We denote the cardinality of a set A by $|A|$. If there is a bijection between two sets, they have the same cardinality. Any set which is bijective to the set $[N]$ has cardinality N.

Remark 1. If there is an injective function $f : B \to A$, then the cardinality of B is at most that of A, i.e. $|B| \leqslant |A|$

Example 1.9. The cardinality of $A = \{a,b,c\}$ is $|A| = 3$, since there is a bijection between $B = \{1,2,3\}$ and $A = \{a,b,c\}$.

Definition 1.10 (Countable). Any set which is bijective to a subset of natural numbers $\mathbb{N}$ is called a **countable** set. Any set which has a finite cardinality is called a **countably finite** set. Any set which is bijective to the set of natural numbers $\mathbb{N}$ is called a **countably infinite** set.

Example 1.11 (Number of functions). Let B and A be finite sets with cardinalities $n_1 \triangleq |B|, n_2 \triangleq |A|$ respectively, then the following are true.
(a) The number of functions $\left|A^B\right| = |A|^{|B|} = n_2^{n_1}$.
(b) If $n_1 = n_2 = n$, the number of bijective function from $B \to A$ is $n!$.
(c) If $n_2 \geqslant n_1$, then the number of injective functions is $\frac{n_2!}{(n_2-n_1)!}$.
(d) If $n_2 \leqslant n_1$, then the number of surjective functions is ?.

Exercise 1.12. Show the following are true.

1. $\left|A^B\right| = |A|^{|B|}$.
2. $A^{[N]}$ is set of all A-valued N-length sequences.
3. $A^{\mathbb{N}}$ is a set of all A-valued countably infinite sequences indexed by the set of natural numbers $\mathbb{N}$.
4. The sets $\mathbb{N}, \mathbb{Z}_+, \mathbb{Z}, \mathbb{Q}$ have the same cardinality.