1. For some index sets $T, I \subset \mathbb{N}$, let $\mathcal{F}_i \triangleq \{\mathcal{F}_t^i : t \in T\}$ be a filtration on measurable space $(\Omega, \mathcal{F})$. Check whether $\mathcal{F}_\bullet \triangleq \{\mathcal{F}_t : t \in T\}$ and $\mathcal{G}_\bullet \triangleq \{\mathcal{G}_t : t \in T\}$ are filtrations or not, for some $\mathcal{F}_t \triangleq \bigcap_{i \in I} \mathcal{F}_t^i$, $\mathcal{G}_t \triangleq \bigcup_{i \in I} \mathcal{F}_t^i$ respectively.

2. Let $\mathcal{F}_\bullet \triangleq \{\mathcal{F}_t : t \in T\}$ for some index set T, be a filtration on space $(\Omega, \mathcal{F})$. Let $\tau_1, \tau_2 : \Omega \to \mathbb{N}^2$ be two stopping times Check whether the following are stopping times or not.
   (a) $\min\{\tau_1, \tau_2\}$
   (b) $\max\{\tau_1, \tau_2\}$
   (c) $\tau_1 + \tau_2$

3. Let index set $T \triangleq [0, \infty)$. Let $\mathcal{F}_\bullet \triangleq \{\mathcal{F}_t : t \in T\}$ is a filtration on $(\Omega, \mathcal{F})$. If $(\tau_n : n \in \mathbb{N})$ is a sequence of stopping times, in the above filtration, check whether the following are stopping times or not.
   (a) $\inf\{\tau_n : n \in \mathbb{N}\}$
   (b) $\sup\{\tau_n : n \in \mathbb{N}\}$
   (c) $\liminf_{n\to\infty} \tau_n$
   (d) $\limsup_{n\to\infty} \tau_n$

4. Let $T \subseteq \mathbb{R}_+$ be an index set. A process X defined on probability space $(\Omega, \mathcal{F}, P)$ is said to be adapted to the filtration $\mathcal{F}_\bullet \triangleq (\mathcal{F}_t : t \in \mathbb{R}_+)$, if $\sigma(X_t) \subseteq \mathcal{F}_t$ for each $t \in T$.
   (a) When $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a sequence of the indicator random variables such that $X_t(\omega) \triangleq \mathbb{1}_{\{[a,b]\}}(\omega)$ for each $\omega \in \Omega$ and some $a < b \in \mathbb{R}_+$ and $\mathcal{F}_\bullet$ is the filtration spanned by the sequence of borel σ- algebras, i.e., $\mathcal{F}_n \triangleq \sigma(\{(-\infty, x] : x \in \mathbb{R}, k \in \mathbb{N}\})$, check whether X is adapted to the filtration $\mathcal{F}_\bullet$.
   (b) X is adapted to its natural filtration. Also, argue why does this hold true for all stochastic processes defined on $(\Omega, \mathcal{F}, P)$.

5. For some $T \in \mathbb{N}$, let $X : \Omega \to \mathbb{R}_+^T$ be a random process defined on $(\Omega, \mathcal{F}, P)$, adapted to the filtration $\mathcal{F}_\bullet$. Let $\tau : \Omega \to \mathbb{N}$ be the stopping time with respect to $\mathcal{F}_\bullet$. Show that the stopped process X^τ is adapted to the filtration $\mathcal{F}_\bullet$.

6. For some $T \subseteq \mathbb{R}_+$, let $X : \Omega \to \mathbb{R}^T$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$. Show that X follows strong Markov property.