1. Let $X : \Omega \to \{0,1\}^{\mathbb{Z}_+}$ be a homogenoeus markov chain with transition probabilities $p_{00} = 1-p, p_{01} = p, p_{10} = 1-q, p_{11} = q$ for some $p \neq q \in (0,1)$. Let $\tau : \Omega \to \mathbb{N}$ be such that τ is almost surely finite and defined as $\tau \triangleq \inf\{n > 0 : X_n = 1\}$.
   (a) Prove that τ is a stopping time adapted to the natural filtration $\mathcal{F}_\bullet \triangleq (\mathcal{F}_n : n \in \mathbb{N})$.
   (b) Find $\mathbb{E}[\tau \mid \{X_0 = 0\}]$.

2. An urn always contains 2 balls, red and blue. At each stage, a ball is randomly chosen and then replaced by a new ball, which with probability 0.8 is the same color and with the probability 0.2 is the opposite color, as the ball it replaces. If initially, both the balls are red, then find the probability that the fifth ball selected is red.

3. Consider a finite time markov chain $X : \Omega \to \{0,1,2\}^{\mathbb{Z}_+}$ with the transition probability matrix,

$$\begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

   Verify the following:
   (a) X is irreducible, aperiodic and positive recurrent.
   (b) Find the long-run proportion of the states $x \in \{0,1,2\}$.

4. Consider a gambler who at each play of the game has probability p of winning 1 unit and probability $q = 1-p$ of losing 1 unit. The gambler quits playing either when he has zero or N units. Assume the successive plays of the game are independent.
   (a) Let X_n denote the player's fortune at time n. Argue that $\{X_n, n \in 0,1,2,\dots\}$ is a Markov chain, find its transition probabilities and identify the set of recurrent and transient states.
   (b) Let f_i denote the probability that, starting with i, $0 < i < N$, the gambler's fortune will eventually reach N. Show that $f_i = pf_{i+1} + qf_{i-1}$ and

$$f_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq q \\ \frac{i}{N}, & \text{if } p = q \end{cases}$$

   Hint: Condition on the outcome of the initial play of the game. To solve for f_i consider solutions of the form $f_i = A + B\lambda^i$.
   (c) What is the expected number of plays that the gambler, starting with i units, makes before reaching either zero or N.
   Hint: Apply Wald's Lemma.

5. Let $\zeta : \Omega \to \{0,1\}^{\mathbb{Z}_+}$ be an *i.i.d.* sequence with mean $\mu \triangleq \mathbb{E}\zeta_1 = 0.5$ Let $X : \Omega \to \{0,1\}^{2\mathbb{Z}_+}$ defined as $X_n \triangleq (\zeta_n, \zeta_{n+1})$. Then, is X a discrete time markov process? If yes, write its transition probability matrix P and compute P^2.

6. In a sequence of independent coin tosses, let N denote the number of flips until there is a row of two consecutive heads. Find,
   (a) $P\{N \leqslant 3\}$
   (b) $P\{N = 3\}$.