Lecture-06: Strong Markov Property

1 Strong Markov property

We will consider real valued processes $X : \Omega \to \mathcal{X}^T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with state space $\mathcal{X} \subseteq \mathbb{R}$ and ordered index set $T \subseteq \mathbb{R}$, adapted to its natural filtration by $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$, where $\mathcal{F}_t \triangleq \sigma(X_s, s \leqslant t)$ for all $t \in T$.

Definition 1.1. A process $X : \Omega \to \mathcal{X}^T$ adapted to its natural filtration $\mathcal{F}_\bullet$, is called **Markov** if we have for $t \geqslant s$

$$\mathbb{E}[\mathbb{1}_{\{X_t \leqslant x\}} \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{1}_{\{X_t \leqslant x\}} \mid \sigma(X_s)].$$

Example 1.2. An independent process is trivially Markov, since

$$\mathbb{E}[\mathbb{1}_{\{X_t \leqslant x\}} \mid \mathcal{F}_s] = \mathbb{E}\mathbb{1}_{\{X_t \leqslant x\}} = \mathbb{E}[\mathbb{1}_{\{X_t \leqslant x\}} \mid \sigma(X_s)].$$

Example 1.3. Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined in term of independent step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ as $S_n \triangleq \sum_{i=1}^n X_i$ for all $n \in \mathbb{N}$. The random walk S is Markov with respect to its natural filtration $\mathcal{F}_\bullet$. To see this, we take $n \in \mathbb{N}$, denote the distribution function for X_{n+1} as $F_{X_{n+1}} : \mathbb{R} \to [0,1]$, and observe from the independence of X_{n+1} and $\mathcal{F}_n$ that

$$\mathbb{E}[\mathbb{1}_{\{S_{n+1} \leqslant x\}} \mid \mathcal{F}_n] = F_{X_{n+1}}(x - S_n) = \mathbb{E}[\mathbb{1}_{\{X_{n+1} \leqslant x - S_n\}} \mid \sigma(S_n)] = \mathbb{E}[\mathbb{1}_{\{S_{n+1} \leqslant x\}} \mid \sigma(S_n)].$$

Definition 1.4. Let $X : \Omega \to \mathcal{X}^T$ be a real valued Markov process adapted to its natural filtration $\mathcal{F}_\bullet$. Let τ be a stopping time with respect to this filtration, then the process X is called **strongly Markov** if for all $x \in \mathbb{R}$ and $t > 0$, we have

$$\mathbb{E}[\mathbb{1}_{\{X_{t+\tau} \leqslant x\}} \mid \mathcal{F}_\tau] = \mathbb{E}[\mathbb{1}_{\{X_{t+\tau} \leqslant x\}} \mid \sigma(X_\tau)]. \tag{1}$$

Exercise 1.5. Consider a random process $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ adapted to its natural filtration $\mathcal{F}_\bullet$, a stopping time $\tau : \Omega \to I \subseteq \mathbb{R}$ adapted to $\mathcal{F}_\bullet$ and a random variable $Y : \Omega \to \mathbb{R}$ all defined on the same probability space $(\Omega, \mathcal{F}, P)$. If I is countable, then show that

$$\mathbb{E}[Y \mid \sigma(X_\tau)] = \sum_{i \in I} \mathbb{1}_{\{\tau = i\}} \mathbb{E}[Y \mid \sigma(X_i, \{\tau = i\})].$$

From the almost sure uniqueness of conditional expectation, it suffices to show that the right hand side of the above equation is $\sigma(X_\tau)$ measurable, absolutely integrable, and orthogonal. For orthogonality, one needs to show that for any $A \in \sigma(X_\tau)$,

$$\mathbb{E}[\mathbb{1}_A Y] = \mathbb{E}[\mathbb{1}_A \sum_{i \in I} \mathbb{1}_{\{\tau = i\}} \mathbb{E}[Y \mid \sigma(X_i, \{\tau = i\})]].$$

Lemma 1.6. *Consider a Markov process $X : \Omega \to \mathcal{X}^T$ adapted to its natural filtration $\mathcal{F}_\bullet$. and a stopping time τ with respect to $\mathcal{F}_\bullet$. If the stopping time τ is almost surely countable, then the process X is strongly Markov at τ.*

Proof. Let $I \subseteq T$ be the countable set such that $P\{\tau \notin I\} = 0$. We will show that the right hand side of (1) satisfies measurability, integrability, and orthogonality of conditional expectation $\mathbb{E}[\mathbb{1}_{\{X_{t+\tau} \leqslant x\}} \mid \mathcal{F}_\tau]$, and the result follows from the a.s. uniqueness of conditional expectation.

Measurability: Recall $\sigma(X_\tau) \subseteq \sigma(X^\tau) \subseteq \mathcal{F}_\tau$, and since the conditional expectation $\mathbb{E}[\mathbb{1}_{\{X_{\tau+t} \leqslant x\}} \mid \sigma(X_\tau)]$ is $\sigma(X_\tau)$ measurable, it is $\mathcal{F}_\tau$ measurable.

Integrability: Since $0 \leqslant \mathbb{1}_{\{X_{\tau+t} \leqslant x\}} \leqslant 1$, from the monotonicity of the conditional expectation it follows that $0 \leqslant \mathbb{E}[\mathbb{1}_{\{X_{\tau+t} \leqslant x\}} \mid \sigma(X_\tau)] \leqslant 1$, and hence it is absolutely integrable.

Orthogonality: Fix $A \in \mathcal{F}_\tau$. It suffices to show that for all $x \in \mathbb{R}$ and $t > 0$,

$$\mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{\{X_{\tau+t} \leqslant x\}} | \sigma(X_\tau)]] = \mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{X_{\tau+t} \leqslant x\}}].$$

From almost sure countability of τ, we can write $A = \cup_{i \in I} A \cap \{\tau = i\}$, where $A \cap \{\tau = i\} \in \mathcal{F}_i$ for all $i \in I$. From the tower property of conditional expectation and $\mathcal{F}_i$-measurability of $A \cap \{\tau = i\}$,

$$\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{X_{t+\tau} \leqslant x\}}] = \sum_{i \in I} \mathbb{E}[\mathbb{1}_{A \cap \{X_{t+\tau} \leqslant x\} \cap \{\tau = i\}}] = \sum_{i \in I} \mathbb{E}[\mathbb{E}[\mathbb{1}_{A \cap \{X_{t+i} \leqslant x\} \cap \{\tau = i\}} | \mathcal{F}_i]] = \sum_{i \in I} \mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau = i\}} \mathbb{E}[\mathbb{1}_{\{X_{t+i} \leqslant x\}} | \mathcal{F}_i]].$$

From Markov property of process X, we have $\mathbb{E}[\mathbb{1}_{\{X_{t+i} \leqslant x\}} \mid \mathcal{F}_i] = \mathbb{E}[\mathbb{1}_{\{X_{t+i} \leqslant x\}} \mid \sigma(X_i)]$. This result together with Exercise 1.5, we have

$$\sum_{i \in I} \mathbb{1}_{\{\tau = i\}} \mathbb{E}[\mathbb{1}_{\{X_{t+i} \leqslant x\}} \mid \sigma(X_i)] = \sum_{i \in I} \mathbb{1}_{\{\tau = i\}} \mathbb{E}[\mathbb{1}_{\{X_{t+\tau} \leqslant x\}} \mid \sigma(X_i)] = \mathbb{E}[\mathbb{1}_{\{X_{t+\tau} \leqslant x\}} \mid \sigma(X_\tau)].$$

The result follows from the linearity of expectation.

□

Corollary 1.7. *Any Markov process on countable index set T is strongly Markov.*

Proof. For a countable index set T, any associated stopping time is countable. □

Corollary 1.8. *Let τ be a stopping time with respect to the natural filtration $\mathcal{F}_\bullet$ of an* i.i.d. *random sequence X. Then $(X_{\tau+1}, \dots, X_{\tau+n})$ is independent of $\mathcal{F}_\tau$ for each $n \in N$ and identically distributed to $(X_1, \dots, X_n)$.*

Proof. Let $F : \mathbb{R} \to [0,1]$ be the common distribution for the *i.i.d.* sequence X, then it suffices to show that

$$\mathbb{E}\left[\prod_{i=1}^{n} \mathbb{1}_{\{X_{\tau+i} \leqslant x_i\}} \mid \mathcal{F}_\tau\right] = \prod_{i=1}^{n} F(x_i), \quad x \in \mathbb{R}^n.$$

Since RHS of the above equation is a constant in $[0,1]$, the measurability and integrability are clear. To show orthogonality, we fix $A \in \mathcal{F}_\tau$ and we need to show that

$$\mathbb{E}[\mathbb{1}_A \prod_{i=1}^{n} F(x_i)] = \mathbb{E}[\mathbb{1}_A \prod_{i=1}^{n} \mathbb{1}_{\{X_{\tau+i} \leqslant x_i\}}].$$

We can write $\mathbb{1}_A = \sum_{m \in \mathbb{N}} \mathbb{1}_A \mathbb{1}_{\{\tau = m\}}$ where $A \cap \{\tau = m\} \in \mathcal{F}_m$. Therefore, from the linearity of expectation, the tower property of conditional expectation, and from X being *i.i.d.* , we can write

$$\mathbb{E}[\mathbb{1}_A \prod_{i=1}^{n} \mathbb{1}_{\{X_{\tau+i} \leqslant x_i\}}] = \sum_{m \in \mathbb{N}} \mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau = m\}} \mathbb{E}[\prod_{i=1}^{n} \mathbb{1}_{\{X_{m+i} \leqslant x_i\}} \mid \mathcal{F}_m]] = \mathbb{E}[\mathbb{1}_A \sum_{m \in \mathbb{N}} \mathbb{1}_{\{\tau = m\}} \prod_{i=1}^{n} F(x_i)] = \mathbb{E}[\mathbb{1}_A \prod_{i=1}^{n} F(x_i)].$$

□

Theorem 1.9. *Let $X : \Omega \to \mathcal{X}^T$ be any real-valued Markov process adapted to its natural filtration $\mathcal{F}_\bullet$, with right-continuous sample paths. If the map $t \mapsto \mathbb{E}[f(X_s) \mid \sigma(X_t)]$ is right-continuous for each bounded continuous function f, then X is strongly Markov.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, $t \geqslant 0$, and τ be an $\mathcal{F}_\bullet$-adapted stopping time. It suffices to show that $f(X_t)$ satisfies the strong Markov property. For each $m \in \mathbb{N}$, consider the intervals $I_{k,m} \triangleq ((k-1)2^{-m}, k2^{-m}]$ for all $k \in [2^{2m}]$, and define

$$\tau_m \triangleq \sum_{k=1}^{2^{2m}} k2^{-m} \mathbb{1}_{\{\tau \in I_{k,m}\}}.$$

We observe that τ_m is adapted to $\mathcal{F}_\bullet$ and takes countable values for each m. Further, we have $\tau \mathbb{1}_{\{\tau \leqslant 2^m\}} \leqslant \tau_m \leqslant 2^m$ and τ_m is decreasing in m. From a.s. finiteness of stopping time τ, for almost all outcomes $\omega \in \Omega$

there exists an $m_0(\omega) \in \mathbb{N}$ such that $\tau \leqslant \tau_m$. Hence, $\tau_m \downarrow \tau$ almost surely. Since $\tau \leqslant \tau_m$, it follows that $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_m}$. From the strong Markov property for the Markov process X at countably valued stopping times, we have

$$\mathbb{E}[f(X_{\tau_m+t}) \mid \mathcal{F}_{\tau_m}] = \mathbb{E}[f(X_{\tau_m+t}) \mid \sigma(X_{\tau_m})].$$

From the orthogonality property of conditional expectation, it follows that for each $A \in \mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_m}$, we have

$$\mathbb{E}[\mathbb{1}_A f(X_{\tau_m+t})] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[f(X_{\tau_m+t}) | \sigma(X_{\tau_m})]].$$

Taking limit as $\tau_m \downarrow \tau$ on both sides and applying dominated convergence theorem, we get

$$\mathbb{E}[\mathbb{1}_A f(X_{\tau+t})] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[f(X_{\tau+t}) | \sigma(X_\tau)]].$$

□

Corollary 1.10. *The counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with the Poisson point process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, satisfies the strong Markov property.*

Proof. It suffices to check the right continuity of the map $t \mapsto \mathbb{E}_{N_t} f(N_s)$ for $s \geqslant t$ and any bounded continuous function f, which holds from the stationary and independent increment property of Poisson process N_t. In particular, $N_s - N_t$ is a Poisson random variable with mean $\Lambda(t,s]$ and independent of N_t, and hence

$$\mathbb{E}_{N_t} f(N_s) = \mathbb{E}_{N_t} f(N_s - N_t + N_t) = \sum_{k \in \mathbb{Z}_+} e^{-\Lambda(t,s]} \frac{\Lambda(t,s]^k}{k!} f(N_t + k).$$

The continuity of the map follows from the right continuity of N_t, boundedness and continuity of f, continuity of $\Lambda(t, t+s]$, and bounded convergence theorem. □

Corollary 1.11. *The standard Brownian motion $B : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ satisfies the strong Markov property.*

Proof. It suffices to check the right continuity of the map $t \mapsto \mathbb{E}_{B_t} f(B_s)$ for $s \geqslant t$ and any bounded continuous function f, which holds from the stationary and independent increment property of Brownian motion B_t. In particular, $B_s - B_t$ is a Gaussian random variable with zero mean and variance s, independent of B_t. Therefore,

$$\mathbb{E}_{B_t} f(B_s) = \mathbb{E}_{B_t} f(B_s - B_t + B_t) = \int_{x \in \mathbb{R}} e^{-\frac{x^2}{2(s-t)}} f(B_t + x) dx.$$

The continuity of the map follows from the continuity of B_t, boundedness and continuity of f, and bounded convergence theorem. □

Definition 1.12. Let $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be a one-dimensional random walk associated with an *i.i.d.* positive step-size sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. We define the associated **counting process** $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ such that $N_t \triangleq \sum_{n \in \mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$ is the number of steps in time $(0, t]$.

Proposition 1.13. *Let $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ be the counting process associated with a random walk $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, and $\mathcal{G}_\bullet$ be the natural filtration for the positive step size sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. Then $(N_{S_m+t_1} - N_{S_m}, \dots, N_{S_m+t_n} - N_{S_m})$ is independent of $\mathcal{G}_m$ and has the same joint distribution as $(N_{t_1}, \dots, N_{t_n})$.*

Proof. Recall that $\{N_t = k\} = \{S_k \leqslant t, S_{k+1} > t\}$, and hence we can write

$$\{N_{S_m+t} - N_{S_m} = k\} = \{S_{m+k} \leqslant S_m + t, S_{m+k+1} > S_m + t\}.$$

Since $S_{m+k} - S_m$ has same distribution as S_k for all $k \geqslant 0$ and is independent of $\mathcal{G}_m$, we can write

$$P(\bigcap_{i=1}^n \{N_{S_m+t_i} - N_{S_m} = k_i\} | \mathcal{G}_m) = P(\bigcap_{i=1}^n \{S_{k_i-m} \leqslant t_1, S_{k_i-m} > t_i\}) = P(\bigcap_{i=1}^n \{N_{t_i} = k_i\}).$$

□