Lecture-07: Renewal Process

1 Counting processes

Definition 1.1. A right continuous stochastic process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a **counting process** if (a) $N_0 = 0$ and (b) the map $t \mapsto N_t$ is non-decreasing for each outcome $\omega \in \Omega$.

Lemma 1.2. *A counting process has finitely many jumps in a finite interval* $(0, t]$.

Definition 1.3. A counting process is called **simple** if it has discontinuities of unit size.

Definition 1.4. The nth point of discontinuity of a simple counting process N is called the n**th arrival instant** and is denoted by $S_n : \Omega \to \mathbb{R}_+$ such that $S_0 \triangleq 0$ and $S_n \triangleq \inf\{t \geqslant 0 : N_t \geqslant n\}$ for all $n \in \mathbb{N}$. The random sequence of arrival instants is denoted by $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$.

Remark 1. The arrival sequence S is non-decreasing for each outcome $\omega \in \Omega$, since inf is non-decreasing for decreasing sets. That is for any $n \in \mathbb{N}$, we have $\{t \in \mathbb{R}_+ : N_t \geqslant n+1\} \subseteq \{t \in \mathbb{R}_+ : N_t \geqslant n\}$ from monotonicity of counting processN_t, and thus $S_n \leqslant S_{n+1}$ for all $n \in \mathbb{N}$ from monotonicity of inf.

Definition 1.5. The **inter arrival time** between $(n-1)$th and nth arrival is denoted by $X_n \triangleq S_n - S_{n-1}$. The random sequence of inter arrival times is denoted by $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$.

Exercise 1.6. Show that $P\{X_n \leqslant 0\} = 0$ for simple counting processes.

Lemma 1.7 (Inverse processes). *Inverse of a simple counting process N is its corresponding arrival process S. That is,*

$$\{S_n \leqslant t\} = \{N_t \geqslant n\}. \tag{1}$$

Proof. Let $\omega \in \{S_n \leqslant t\}$. Since N is a non-decreasing process, we have $N_t \geqslant N_{S_n} = n$. Conversely, let $\omega \in \{N_t \geqslant n\}$, then it follows from definition that $S_n(\omega) \leqslant t$. □

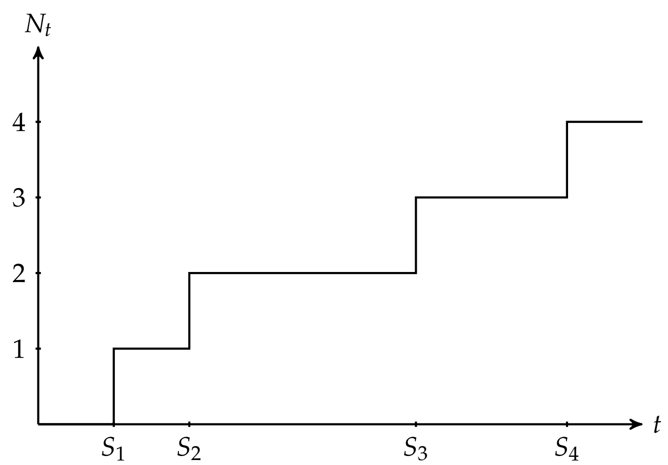

Figure 1: Sample path of a simple counting process.

Remark 2. Let $\mathcal{F}_\bullet = (\mathcal{F}_s : s \geqslant 0)$ be the natural filtration associated with the counting process N, that is $\mathcal{F}_t \triangleq \sigma(N_s, s \in [0,t])$. From (3), we have $\{S_n \leqslant t\} \in \sigma(N_t) \subseteq \mathcal{F}_t$ for all $t \in \mathbb{R}_+$. It follows that $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a sequence of random times adapted to filtration $\mathcal{F}_\bullet$.

Corollary 1.8. *The probability mass function for the counting process N sampled at time t can be written in terms of distribution functions of arrival sequence S as*

$$P\{N_t = n\} = F_{S_n}(t) - F_{S_{n+1}}(t).$$

Proof. The event $\{N_t \geqslant n\}$ is the union of two disjoint events $\{N_t = n\} \cup \{N_t \geqslant n+1\}$, and the result follows from the probability of disjoint unions. □

Definition 1.9. A **point process** is a collection $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ of randomly distributed points, such that $\lim_{n\to\infty} |S_n| = \infty$. A point process is simple if the points are distinct. Let $N(\emptyset) = 0$ and denote the number of points in a measurable set $A \in \mathcal{B}(\mathcal{X})$ by

$$N(A) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \in A\}}.$$

Then $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ is called a **counting process** for the simple point process S.

Remark 3. When $\mathcal{X} = \mathbb{R}_+$, one can order these points of S as an increasing sequence such that $S_n < S_{n+1}$ for all $n \in \mathbb{N}$. Further, we denote the number of points in a half-open interval $(0, t]$ by

$$N_t \triangleq N(0,t] = \sum_{n\in\mathbb{N}} \mathbb{1}_{(0,t]}(S_n) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}.$$

Remark 4. General point processes in higher dimension don't have any inter-arrival time interpretation.

Exercise 1.10. Show that $P\{X_n \leqslant 0\} = 0$ for simple point processes on $\mathbb{R}_+$.

2 Renewal processes

Definition 2.1 (Renewal Instants). Consider an *i.i.d.* sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ of **inter-renewal times** and denote the nth **renewal instant** by $S_n \triangleq \sum_{i=1}^n X_i$ for all $n \in \mathbb{N}$, and $S_0 = 0$. The random sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is called **sequence of renewal instants** or **renewal sequence**.

Remark 5. We interpret X_n as the time between the $(n-1)$th and the nth renewal event, with a common distribution F. If $F(0) = 1$, then it is a trivial process. Hence we will often assume that $F(0) < 1$ to imply a non-degenerate renewal process.

Example 2.2 (Random walk). Random walk S on $\mathbb{R}_+$ with *i.i.d.* non-negative step-sizes $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a renewal sequence.

Example 2.3 (Markov chain). Let $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ be a discrete time homogeneous Markov chain X with state space $\mathcal{X}$. For $X_0 = x \in \mathcal{X}$ and defining $\tau_x^+(0) \triangleq 0$, let the recurrent times be defined inductively as

$$\tau_x^+(n) = \inf\{k > \tau_x^+(n-1) : X_k = x\}. \tag{2}$$

It follows from the strong Markov property of the process X, that $\tau_x^+ : \Omega \to \mathbb{Z}_+^{\mathbb{Z}_+}$ is a renewal sequence.

Definition 2.4 (Renewal process). The associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ that counts number of renewal until time t with *i.i.d.* general inter-renewal times is called a **renewal process**, written as

$$N_t \triangleq \sup\{n \in \mathbb{Z}_+ : S_n \leqslant t\} = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}.$$

Definition 2.5. A renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with *i.i.d.* inter-renewal times sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is said to be **recurrent** if the inter-renewal time X_1 is finite almost surely, the sequence is called **transient** otherwise. A renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is said to be **positive recurrent** if the inter-renewal time X_1 has finite mean.

Remark 6. We will mostly be interested in a positive recurrent renewal process, and hence we will often assume that the mean $\mu = \mathbb{E}X_1 = \int_{\mathbb{R}_+} x dF(x)$ is finite.

Definition 2.6. The renewal process is said to be **periodic** with period d if the *i.i.d.* inter-renewal times $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ take values in a discrete set $\mathcal{X} \subseteq \{nd : n \in \mathbb{Z}_+\}$ and $d = \gcd(\mathcal{X})$ is the largest such number. Otherwise, if there is no such $d > 0$, then the renewal process is said to be **aperiodic**. If the inter-arrival time X_1 is a periodic random variable, then the associated distribution function F is called **lattice**.

Lemma 2.7 (Finiteness). *For a renewal sequence with mean inter-renewal time $\mathbb{E}X_1 > 0$, the number of renewals N_t in the time duration $(0,t]$ is a.s. finite for all $t \in \mathbb{R}_+$.*

Proof. We are interested in knowing the number of renewals per unit time. For each $n \in \mathbb{N}$, the inter-renewal time X_n is non-negative, and hence mean $\mu = \mathbb{E}X_n = \mathbb{E}|X_n|$.

If $\mu = \infty$, then $P\{X_n < \infty\} < 1$ and we define $N \triangleq \inf\{n \in \mathbb{N} : S_n = \infty\}$. It follows that $N_t \leqslant N$ for all times $t \in \mathbb{R}_+$. We further observe that $P\{N = n\} = P\{X_1 < \infty\}^{n-1} P\{X_1 = \infty\}$ and hence $P\{N \in \mathbb{N}\} = 1$, i.e. $S_n = \infty$ almost surely for some finite $n \in \mathbb{N}$. Hence, N_t is almost surely finite for all $t \in \mathbb{R}_+$.

Therefore, we assume that $\mu < \infty$ without any loss of generality. It follows from the L^1 strong law of large numbers that

$$P\left\{\lim_{n\in\mathbb{N}} \frac{S_n}{n} = \mu\right\} = 1.$$

Since $\mu > 0$ from the hypothesis, we must have S_n growing arbitrarily large as n increases. Thus, S_n can be finite for at most finitely many n. Indeed for any finite t, we have the the following set inclusion

$$\bigcap_{n\in\mathbb{N}} \{N_t \geqslant n\} = \bigcap_{n\in\mathbb{N}} \{S_n \leqslant t\} \subseteq \bigcap_{n\in\mathbb{N}} \left\{\frac{S_n}{n} \leqslant \frac{t}{n}\right\} \subseteq \left\{\limsup_{n\in\mathbb{N}} \frac{S_n}{n} = 0\right\}.$$

Since $\mu > 0$, we obtain $\left\{\limsup_{n\in\mathbb{N}} \frac{S_n}{n} = 0\right\} \subseteq \left\{\lim_{n\in\mathbb{N}} \frac{S_n}{n} = \mu\right\}^c$, it follows that $P\{N_t = \infty\} = 0$. □

Remark 7. Since the number of renewals N_t in any finite duration $(0,t]$ is finite for any $t \in \mathbb{R}_+$, we can replace supremum by maximum, and $N_t = \max\{n \in \mathbb{N} : S_n \leqslant t\}$.

Exercise 2.8. Show that for sequences $x \in \mathbb{R}^{\mathbb{N}}$ and $\alpha \in \mathbb{R}^{\mathbb{N}}$, if $x_n \leqslant \alpha_n$ for all $n \in \mathbb{N}$, then $\limsup_n x_n \leqslant \limsup_n \alpha_n$.

2.1 Delayed renewal processes

Many times in practice, we have a *delayed start* to a renewal sequence. That is, the renewal seqence has independent inter renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, where the common distribution for X_n is F when $n \geqslant 2$, and the distribution of first inter-arrival time X_1 is G. Similar to the renewal sequence, the initial renewal instant is assumed to be $S_0 = 0$ and the nth renewal instant is $S_n \triangleq \sum_{i=1}^{n} X_i$ for all $n \in \mathbb{N}$. The associated counting process is called a **delayed renewal process** and denoted by $N^D : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. The following inverse relationship continues to hold between the counting process and the renewal sequence,

$$\left\{N^D(t) \geqslant n\right\} = \{S_n \leqslant t\}. \tag{3}$$

Example 2.9 (Markov chain). Let $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ be a discrete time homogeneous Markov chain. For $X_0 = x \in \mathcal{X}$ and $y \neq x$, we define $\tau_y^+(0) \triangleq 0$. We inductively define the kth visit time to state y as

$$\tau_y^+(k) \triangleq \inf\left\{n > \tau_y^+(k-1) : X_n = y\right\}.$$

It follows from the strong Markov property of the process X, that $\tau_y^+ : \Omega \to \mathbb{Z}_+^{\mathbb{Z}_+}$ is a delayed renewal sequence.

Exercise 2.10. Consider a delayed renewal sequence with positive mean inter-renewal duration $\mathbb{E}X_n > 0$ for $n \geqslant 2$. Show that the number of renewals N_t^D in the time duration $(0,t]$ is almost surely finite for all finite $t \in \mathbb{R}_+$.

A Strong Markov property of renewal process

Proposition A.1. *Let $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ be the renewal process associated with a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. For the inter renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, we define $\mathcal{G}_m \triangleq \sigma(X_1,\dots,X_m)$ for each $m \in \mathbb{N}$ to define its natural filtration $\mathcal{G}_\bullet \triangleq (\mathcal{G}_m : m \in \mathbb{N})$. Then the random vector $(N_{S_m+t_1} - N_{S_m},\dots,N_{S_m+t_n} - N_{S_m})$ is independent of $\mathcal{G}_m$ and has the same joint distribution as $(N_{t_1},\dots,N_{t_n})$.*

Proof. Recall that $\{N_t = k\} = \{S_k \leqslant t, S_{k+1} > t\}$, and hence we can write

$$\{N_{S_m+t} - N_{S_m} = k\} = \{S_{m+k} \leqslant S_m + t < S_{m+k+1}\}.$$

We observe that $\mathcal{G}_m = \sigma(S_1,\dots,S_m)$ since S and X have a bijection. Further, we observe that $S_{m+k} - S_m$ is independent of $\mathcal{G}_m$ and has the same distribution as S_k for all $k \geqslant 0$. Thus, we can write

$$P(\bigcap_{i=1}^{n}\{N_{S_m+t_i} - N_{S_m} = k_i\} \mid \mathcal{G}_m) = P(\bigcap_{i=1}^{n}\{S_{m+k_i} - S_m \leqslant t_i < S_{m+k_i-1} - S_m\} \mid \mathcal{G}_m) = P(\bigcap_{i=1}^{n}\{N_{t_i} = k_i\}).$$

□