Lecture-08: Distribution and renewal functions

1 Convolution of distribution functions

Definition 1.1. For two distribution functions $F, G : \mathbb{R} \to [0,1]$ the convolution of F and G is a distribution function $F * G : \mathbb{R} \to [0,1]$ defined as

$$(F * G)(x) \triangleq \int_{y \in \mathbb{R}} F(x-y) dG(y),\ x \in \mathbb{R}.$$

Lemma 1.2. *Let $F, G \in [0,1]^{\mathbb{R}}$, then the convolution $F * G$ is a distribution function.*

Proof. It suffices to show that the function $(F * G)$ is
(a) right continuous, i.e. $\lim_{x_n \downarrow x} (F * G)(x_n)$ exists,
(b) non-decreasing, i.e. $(F * G)(z) \geqslant (F * G)(x)$ for all $z \geqslant x$,
(c) having left limit of zero and right limit of unity, i.e. $\lim_{x \to -\infty} (F * G)(x) = 0, \lim_{x \to \infty} (F * G)(x) = 1$.
Part (a) and (c) can be verified by exchanging limit and integration using Monotone convergence theorem. Part (b) can be verified from monotonicity of integration. □

Lemma 1.3. *Convolution is a symmetric and bi-linear operator.*

Proof. We note that for any $F, G \in [0,1]^{\mathbb{R}}$, we have $*(F, G) \triangleq F * G$.

(a) **Symmetry.** Let $F, G \in [0,1]^{\mathbb{R}}$ be distribution functions. Then, it suffices to show that $F * G = G * F$. To this end, we observe from Fubini's theorem for nonnegative functions, exchanging order of integration, and change of variables, that

$$\int_{y \in \mathbb{R}} F(x-y) dG(y) = \int_{y \in \mathbb{R}} \int_{u \leqslant x-y} dF(u) dG(y) = \int_{u \in \mathbb{R}} dF(u) \int_{y \leqslant x-u} dG(y) = \int_{u \in \mathbb{R}} dF(u) G(x-u).$$

(b) **Bilinearity.** It suffices to show for any two finite sets of distribution functions $(F_i \in [0,1]^{\mathbb{R}} : i \in [n])$ and $(G_j \in [0,1]^{\mathbb{R}} : j \in [m])$ and vectors $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m$, we have

$$\Big(\sum_{i \in [n]} \alpha_i F_i\Big) * \Big(\sum_{j \in [m]} \beta_j G_j\Big) = \sum_{i \in [n]} \sum_{j \in [m]} \alpha_i \beta_j (F_i * G_j).$$

This follows from the linearity of integration in its arguments.

□

Lemma 1.4. *Let X and Y be two independent random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with distribution functions F and G respectively, then the distribution of $X + Y$ is given by $F * G$.*

Proof. The distribution function of sum $X + Y$ is denoted by $H \in [0,1]^{\mathbb{R}}$ where $H(z) \triangleq \mathbb{E}\mathbb{1}_{\{X+Y \leqslant z\}}$ for any $z \in \mathbb{R}$. It follows from the tower property of conditional expectation and independence of X and Y that

$$H(z) = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{X+Y \leqslant z\}} | \sigma(Y)]] = \mathbb{E}[F(z-Y)] = \int_{y \in \mathbb{R}_+} F(z-y) dG(y).$$

□

Definition 1.5. Consider a real valued random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with *i.i.d.* step size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$, defined as $S_n \triangleq \sum_{i=1}^n X_i$ for all $n \in \mathbb{N}$. We denote the distribution of X_1 by F and the distribution of S_n by F_n, i.e. $F_n(t) \triangleq P\{S_n \leqslant t\}$ for all $t \in \mathbb{R}$.

Remark 1. The distribution F_n is computed inductively as $F_n = F_{n-1} * F$ for all $n \geqslant 2$, where $F_1 = F$.

Remark 2. For a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with *i.i.d.* inter renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having a common distribution $F \in [0,1]^{\mathbb{R}_+}$, the distribution function of the nth renewal instant is the n-fold convolution F_n of the distribution function F.

Example 1.6 (Poisson process). Consider a renewal sequence $S:\Omega\to\mathbb{R}_+^{\mathbb{N}}$ with *i.i.d.* inter renewal times $X:\Omega\to\mathbb{R}_+^{\mathbb{N}}$ having a common exponential distribution $F\in[0,1]^{\mathbb{R}_+}$ defined as $F(x)\triangleq 1-e^{-\lambda x}$ for any $x\in\mathbb{R}_+$ and rate $\lambda\in\mathbb{R}_+$. We will show by induction that the distribution of nth renewal instant at any time $t\in\mathbb{R}_+$ is

$$F_n(t)\triangleq\int_0^t\frac{\lambda^n s^{n-1}}{(n-1)!}e^{-\lambda s}ds.$$

We first observe that the statement holds true for base case of $n=1$, since $F_1(t)=\int_0^t\lambda e^{-\lambda s}ds=1-e^{-\lambda t}=F(t)$ for all $t\in\mathbb{R}_+$. We assume that the hypothesis holds true for step $n-1$, and compute $F_n=F_{n-1}*F$ written as

$$\begin{aligned}F_n(t)&=\int_{s=0}^t F(t-s)dF_{n-1}(s)=\lambda^{n-1}\int_0^t(e^{-\lambda s}-e^{-\lambda t})d\frac{s^{n-1}}{(n-1)!}\\&=(e^{-\lambda s}-e^{-\lambda t})\frac{s^{n-1}}{(n-1)!}\Big|_{s=0}^t+\int_0^t\frac{\lambda^n s^{n-1}}{(n-1)!}e^{-\lambda s}ds.\end{aligned}$$

The equality in the second line of the above equation follows from the integration by parts.

Corollary 1.7. *The distribution function of nth arrival instant S_n for delayed renewal sequence is $G*F_{n-1}$.*

Corollary 1.8. *The distribution function of counting process $N^D:\Omega\to\mathbb{Z}_+^{\mathbb{R}_+}$ for the delayed renewal sequence is*

$$P\left\{N_t^D=n\right\}=P\left\{S_n\leqslant t\right\}-P\left\{S_{n+1}\leqslant t\right\}=(G*F_{n-1})(t)-(G*F_n)(t).$$

2 Renewal functions

Definition 2.1. Mean of the counting process $N:\Omega\to\mathbb{Z}_+^{\mathbb{R}_+}$ is called the **renewal function** denoted by $m:\mathbb{R}_+\to\mathbb{R}_+$ defined by $m_t\triangleq\mathbb{E}[N_t]$ for all $t\in\mathbb{R}_+$.

Proposition 2.2. *Renewal function $m\in\mathbb{R}_+^{\mathbb{R}_+}$ for a renewal process $N:\Omega\to\mathbb{Z}_+^{\mathbb{R}_+}$ is $m_t=\sum_{n\in\mathbb{N}}F_n(t)$ for all $t\in\mathbb{R}_+$, where the distribution of renewal instant S_n is denoted by $F_n\in[0,1]^{\mathbb{R}_+}$ for each $n\in\mathbb{N}$.*

Proof. Using the inverse relationship between counting process and the arrival instants, we can write

$$m_t=\mathbb{E}[N_t]=\sum_{n\in\mathbb{N}}P\left\{N_t\geqslant n\right\}=\sum_{n\in\mathbb{N}}P\left\{S_n\leqslant t\right\}=\sum_{n\in\mathbb{N}}F_n(t).$$

For the second inequality in the above equation, we observe that $\mathbb{E}N_t=\mathbb{E}\sum_{m\in\mathbb{N}}\sum_{n=1}^m\mathbb{1}_{\{N_t=m\}}$. Switching the order of summations and using monotone convergence theorem to exchange expectation and infinite sum, we obtain $\mathbb{E}N_t=\mathbb{E}\sum_{n\in\mathbb{N}}\sum_{m\geqslant n}\mathbb{1}_{\{N_t=m\}}=\mathbb{E}\sum_{n\in\mathbb{N}}\mathbb{1}_{\{N_t\geqslant n\}}=\mathbb{E}\sum_{n\in\mathbb{N}}P\left\{N_t\geqslant n\right\}$. ☐

Example 2.3 (Poisson process). Consider the renewal sequence $S:\Omega\to\mathbb{R}_+^{\mathbb{N}}$ with *i.i.d.* inter renewal times $X:\Omega\to\mathbb{R}_+^{\mathbb{N}}$ having a common exponential distribution $F\in[0,1]^{\mathbb{R}_+}$ defined as $F(x)\triangleq 1-e^{-\lambda x}$ for any $x\in\mathbb{R}_+$ and rate $\lambda\in\mathbb{R}_+$. The associate renewal function with this renewal sequence is

$$m_t=\sum_{n\in\mathbb{N}}F_n(t)=\int_0^t\lambda\Big(e^{-\lambda s}\sum_{n\in\mathbb{Z}_+}\frac{(\lambda s)^n}{n!}\Big)ds=\int_0^t\lambda ds=\lambda t.$$

Corollary 2.4. *The renewal function $m^D:\mathbb{R}_+\to\mathbb{R}_+$ for a delayed renewal process $N^D:\Omega\to\mathbb{Z}_+^{\mathbb{R}_+}$ with distribution $G\in[0,1]^{\mathbb{R}_+}$ for the first inter renewal time X_1 and common distribution $F\in[0,1]^{\mathbb{R}_+}$ for inter renewal times X_n for $n\geqslant 2$, is given by $m^D=G+G*m$.*

Proof. We can write the renewal function for the delayed renewal process as $m_t^D=\mathbb{E}N_t^D=\sum_{n\in\mathbb{N}}(G*F_{n-1})(t)=G(t)+(G*m)(t)$. ☐

Remark 3. If $G=F$, then we have the identity $m=F+F*m$.

3 Laplace transform of distribution functions and renewal functions

Definition 3.1. The Laplace transform $\mathcal{L} : [0,1]^{\mathbb{R}} \to \mathbb{C}^{\mathbb{C}}$ for a distribution function $F \in [0,1]^{\mathbb{R}}$ is a map $\mathcal{L}_F \in \mathbb{C}^{\mathbb{C}}$ defined for all $s \in \mathbb{C}$ such that $|\mathcal{L}_F(s)| < \infty$, as

$$\mathcal{L}_F(s) \triangleq \int_{y\in\mathbb{R}} e^{-sy} dF(y).$$

Remark 4. If $X : \Omega \to \mathbb{R}$ is a random variable with distribution function F, then $\mathcal{L}_F(s) = \mathbb{E}e^{-sX}$.

Lemma 3.2. *The Laplace transform of convolution of two distribution functions is product of Laplace transform of individual distribution functions, i.e. if $F,G \in [0,1]^{\mathbb{R}}$ are distribution functions, then $\mathcal{L}_{F*G} = \mathcal{L}_F\mathcal{L}_G$.*

Proof. Let $F,G \in [0,1]^{\mathbb{R}}$ be two distribution functions, then $d(F*G)(x) = \int_{y\in\mathbb{R}} dF(x-y)dG(y)$ from exchange of limits and integration using monotone convergence theorem. Therefore,

$$\mathcal{L}_{F*G}(s) = \int_{x\in\mathbb{R}} e^{-sx} \int_{y\in\mathbb{R}} dF(x-y)dG(y) = \int_{y\in\mathbb{R}} e^{-sy}dG(y) \int_{x-y\in\mathbb{R}} e^{-s(x-y)}dF(x-y) = \mathcal{L}_F\mathcal{L}_G.$$

Alternatively, consider two independent random variables X,Y with distributions F,G respectively. Then the distribution of $X+Y$ is $F*G$ and $\mathcal{L}_{F*G}(s) = \mathbb{E}e^{-s(X+Y)} = \mathbb{E}e^{-sX}\mathbb{E}e^{-sY} = (\mathcal{L}_F\mathcal{L}_G)(s)$ for any $s \in \mathcal{C}$ such that $|\mathcal{L}_F(s)||\mathcal{L}_G(s)| < \infty$. ☐

Remark 5. Consider a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with *i.i.d.* inter renewal time sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ having a common distribution function F. The Laplace transform of the distribution of nth renewal instant S_n is $\mathcal{L}_{F_n} = \mathcal{L}_F^n$.

Corollary 3.3. *The Laplace transform of the renewal function $m \in \mathbb{R}_+^{\mathbb{R}_+}$ is given by $\mathcal{L}_m = \frac{\mathcal{L}_F}{1-\mathcal{L}_F}$ defined for each $s \in \mathbb{C}$ such that $|\mathcal{L}_F(s)| < 1$.*

Corollary 3.4. *The Laplace transform of the renewal function m^D for the delayed renewal process is $\mathcal{L}_{m^D} = \frac{\mathcal{L}_G}{1-\mathcal{L}_F}$ defined for each $s \in \mathbb{C}$ such that $|\mathcal{L}_F(s)| < 1$.*

Example 3.5 (Poisson process). The Laplace transform of an exponential distribution $F \in [0,1]^{\mathbb{R}_+}$ defined as $F(x) \triangleq 1 - e^{-\lambda x}$ for $x \in \mathbb{R}_+$ and rate $\lambda \in \mathbb{R}_+$ is given by $\mathcal{L}_F(s) = \frac{\lambda}{\lambda+s}$ for $\Re(s) > -\lambda$. Consider a renewal sequence S with *i.i.d.* inter renewal times having the common exponential distribution F. The Laplace transform for the distribution F_n of nth renewal instant is given by $\mathcal{L}_{F_n}(s) = \left(1+\frac{s}{\lambda}\right)^{-n}$ for all $s \in \mathbb{C}$ such that $-\Re(s) < \lambda$. The Laplace transform for the renewal function for renewal sequence S is

$$\mathcal{L}_m(s) = \frac{\mathcal{L}_F(s)}{1-\mathcal{L}_F(s)} = \frac{\lambda}{s} \text{ for all } s \in \{r \in \mathbb{C} : -\Re(r) < \lambda < |\lambda + r|\}.$$

That is, if $s = \sigma + j\omega$ for $\sigma,\omega \in \mathbb{R}$, then $-\sigma < \lambda$ and $\lambda^2 < (\lambda+\sigma)^2 + \omega^2$. We observe that it suffices that $\sigma > 0$ or $\omega > \lambda$.

Exercise 3.6. Invert the Laplace transform $\mathcal{L}_{F_n}(s) = \left(1+\frac{s}{\lambda}\right)^{-n}$ in the region of convergence $\{s \in \mathbb{C} : -\Re(r) < \lambda < |\lambda + s|\}$ to obtain the distribution function F_n for the nth arrival instant of a Poisson process with rate λ.

Proposition 3.7. *For renewal sequence S with* i.i.d. *inter renewal times having positive common mean $\mathbb{E}X_1 > 0$, the renewal function is bounded for all finite times.*

Proof. Since $\mathbb{E}X_1 > 0$ and $X_1 \geqslant 0$, it follows that $P\{X_1 = 0\} < 1$. From the continuity of probability, there exists $\alpha > 0$ and $\beta \in (0,1)$, such that $P\{X_n \geqslant \alpha\} = \beta$. We define a map $g_\alpha : \mathbb{R}_+ \to \{0,\alpha\}$ such that $g_\alpha(x) \triangleq \alpha \mathbb{1}_{\{x\geqslant\alpha\}}$ for all $x \in \mathbb{R}_+$. We observe that $g_\alpha(x) \leqslant x$. We can define bivariate random sequence $\bar{X} : \Omega \to \{0,\alpha\}^{\mathbb{N}}$ where $\bar{X}_n \triangleq g_\alpha(X_n)$ for all $n \in \mathbb{N}$. It follows that $\bar{X}$ is *i.i.d.* with probability mass function $P\{\bar{X}_1 = 0\} = P\{X_1 < \alpha\} = 1-\beta$ and $P\{\bar{X}_1 = \alpha\} = \beta$. Further, since $g_\alpha(x) \leqslant x$, it follows that

$\bar{X}_n \leqslant X_n$. It follows that $\bar{S}_n \triangleq \sum_{i=1}^n \bar{X}_i \leqslant \sum_{i=1}^n X_i = S_n$, and hence $\{S_n \leqslant t\} \subseteq \{\bar{S}_n \leqslant t\}$ for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. Let $\bar{N} : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ denote the renewal counting process with *i.i.d.* inter arrival time sequence $\bar{X} : \Omega \to \{0, \alpha\}^{\mathbb{N}}$ with arrivals $\bar{S} : \Omega \to \{d\alpha : d \in \mathbb{Z}_+\}^{\mathbb{N}}$ at integer multiples of α. Then, it follows that for all sample paths and all times $t \in \mathbb{R}_+$,

$$N_t = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}} \leqslant \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\bar{S}_n \leqslant t\}} = \bar{N}_t.$$

Hence, it follows that $m_t \triangleq \mathbb{E}N_t \leqslant \mathbb{E}\bar{N}_t$, and to show finiteness of m_t it suffices to show that renewal function $\bar{m}_t \triangleq \mathbb{E}\bar{N}_t$ associated with quantized inter renewal time is finite at any time $t \in \mathbb{R}_+$.

We observe that $\{\bar{N}_0 = n_0\} = \{\bar{N}_0 = n_0, \bar{N}_\alpha \geqslant 1\} = \cap_{i=1}^{n_0} \{\bar{X}_i = 0\} \cap \{\bar{X}_{n_0+1} = \alpha\}$. We define $M_i \triangleq \sum_{j=0}^{i} n_j$ and observe that $\{\bar{S}_n - \bar{S}_k = 0\} = \cap_{i=k+1}^{n} \{\bar{X}_i = 0\}$. Hence, we write

$$\begin{aligned}
&\{\bar{N}_0 = n_0, \bar{N}_\alpha = n_1\} = \{\bar{N}_0 = n_0, \bar{N}_\alpha = n_1, \bar{N}_{2\alpha} \geqslant 1\} = \{\bar{S}_{M_0} = 0, \bar{S}_{M_0+1} = \alpha\} \cap \{\bar{S}_{M_1} = \alpha, \bar{S}_{M_1+1} = 2\alpha\} \\
&= \bigcap_{i=1}^{M_0} \{\bar{X}_i = 0\} \cap \{\bar{X}_{M_0+1} = \alpha\} \bigcap_{i=M_0+2}^{M_1} \{\bar{X}_i = 0\} \cap \{\bar{X}_{M_1+1} = \alpha\}.
\end{aligned}$$

We can write the joint event of number of arrivals n_i at each arrival instant in $i\alpha$ for $i \in \{0, \dots, k-1\}$, as

$$\begin{aligned}
\bigcap_{i=0}^{k-1} \{\bar{N}_{i\alpha} = n_i\} &= \bigcap_{i=0}^{k-1} \{\bar{N}_{i\alpha} = n_i\} \cap \{\bar{N}_{k\alpha} \geqslant 1\} = \bigcap_{i=0}^{k-1} \{\bar{S}_{M_i} = i\alpha, \bar{S}_{M_i+1} = (i+1)\alpha\} \\
&= \bigcap_{i=1}^{M_0} \{\bar{X}_i = 0\} \bigcap \{\bar{X}_{M_0+1} = \alpha\} \bigcap_{j=1}^{k-1} \Big(\bigcap_{i=M_{j-1}+2}^{M_j} \{\bar{X}_i = 0\} \cap \Big\{\bar{X}_{M_j+1} = \alpha\Big\} \Big).
\end{aligned}$$

It follows that the joint distribution of number of arrivals at first k arrival instants is

$$P\left(\bigcap_{i=0}^{k-1} \{\bar{N}_{i\alpha} = n_i\}\right) = (1-\beta) \prod_{i=0}^{k-1} (\beta)(1-\beta)^{n_i - 1}.$$

It follows that the number of arrivals is independent at each arrival instant $k\alpha$ and geometrically distributed over $\mathbb{N}$ with mean $1/\beta$ for $k \in \mathbb{N}$ and over $\mathbb{Z}_+$ with mean $(1-\beta)/\beta$ for $k = 0$ respectively. Thus, for all $t \geqslant 0$,

$$\mathbb{E}N_t \leqslant \mathbb{E}\bar{N}_t \leqslant \frac{\lceil \frac{t}{\alpha} \rceil}{\beta} \leqslant \frac{\frac{t}{\alpha} + 1}{\beta} < \infty.$$

□

Corollary 3.8. *For delayed renewal sequence with $\mathbb{E}X_2 > 0$, the renewal function is bounded at all finite times.*