Lecture-10: Regenerative Processes

1 Regenerative processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be a renewal sequence, with the associated inter renewal sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. That is, the nth renewal instant is $S_n \triangleq \sum_{i=1}^{n} X_i$ for each $n \in \mathbb{N}$ and the number of renewals is $N_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$ until each time $t \in \mathbb{R}_+$.

Definition 1.1. Consider a stochastic process $Z : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ defined over the same probability space. The nth segment of the joint process $(N, Z) : \Omega \to (\mathbb{Z}_+ \times \mathbb{R})^{\mathbb{R}_+}$ is defined as the sample path in the nth inter renewal duration, written $\zeta_n \triangleq (X_n, (Z_{S_{n-1}+t} : t \in [0, X_n)), \qquad n \in \mathbb{N}$.

Definition 1.2. The process Z is regenerative over the renewal sequence S, if its segments $(\zeta_n : n \in \mathbb{N})$ are *i.i.d.* . The process Z is delayed regenerative, if S is a delayed renewal sequence and the segments $(\zeta_n : n \in \mathbb{N})$ of the joint process are independent with $(\zeta_n : n \geqslant 2)$ being identically distributed.

Definition 1.3. Let $\mathcal{F}_t \triangleq \sigma(N_u, Z_u, u \leqslant t)$ be the history of the regenerative process until time $t \in \mathbb{R}_+$. The renewal sequence S is the **regeneration times** for the process Z, and the process Z possesses the **regenerative property** of the process $(Z_{S_{n-1}+t} : t \geqslant 0)$ being independent of history $\mathcal{F}_{S_{n-1}}$ and distributed identically to Z.

Remark 1. The definition says that probability law is independent of the past and shift invariant at renewal times. That is after each renewal instant, the process becomes an independent probabilistic replica of the process starting from zero.

Remark 2. If the stochastic process Z is bounded, then for any Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, we have
$$\mathbb{E}[f(Z_t) \mid \mathcal{F}_{S_{n-1}}] = \mathbb{E}[f(Z_{t-S_{n-1}}) \mid \sigma(S_{n-1})]\mathbb{1}_{\{t \geqslant S_{n-1}\}} + f(Z_t)\mathbb{1}_{\{t < S_{n-1}\}}.$$

Example 1.4 (Age process). Let $N : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ be the renewal counting process for the renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, then the age at time t is defined as $A_t \triangleq t - S_{N_t}$. We observe that the sample path of age in nth renewal interval is given by
$$A_{S_{n-1}+t} = t, \qquad t \in [0, X_n).$$
Since the segments $(X_n, (t : t \in [0, X_n)))$ are *i.i.d.* , it follows that the age process $A : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ is regenerative.

Example 1.5 (Markov chains). For a discrete time homogeneous, irreducible, and positive recurrent Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ on finite state space $\mathcal{X} \subset \mathbb{R}$, we can inductively define the recurrent times for state $y \in \mathcal{X}$ as $\tau_y^+(0) \triangleq 0$, and
$$\tau_y^+(n) \triangleq \inf\left\{k > \tau_y^+(n-1) : X_k = y\right\}.$$
From the strong Markov property of Markov chain X, it follows that $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ is a delayed renewal sequence. For all $n \in \mathbb{N}$, we define the nth excursion time to the state y as $I_n \triangleq \left\{\tau_y^+(n-1)+1, \dots, \tau_y^+(n)\right\}$ and length of this excursion as $T_y(n) \triangleq \tau_y^+(n) - \tau_y^+(n-1)$. We can write the nth segment for the Markov chain X as $\zeta_n = (T_y(n), (X_{\tau_y^+(n-1)+k} : k \in [T_y(n)])$. Independence of the segments follows from the strong Markov property. Further, in the segment $n \geqslant 2$

of the joint process, we can write the joint distribution for $(T_y(n), X_{\tau_y^+(n-1)+k})$ for $k < T_y(n)$ and $z \neq y$ as

$$P\left\{k < T_y(n) = m, X_{\tau_y^+(n-1)+k} = z\right\} = P_y\left\{\tau_y^+(1) > k, X_k = z\right\} P_z\left\{\tau_y^+(1) = m - k\right\}.$$

The equality follows from the strong Markov property and the homogeneity of process X. It follows that the Markov process X is a delayed regenerative process over delayed renewal sequence τ_y^+.

Example 1.6 (Alternating renewal processes). An *i.i.d.* random sequence $(Z,Y) : \Omega \to (\mathbb{R}_+^2)^{\mathbb{N}}$ is called an **alternating renewal sequence**, where Z_n and Y_n are nth **on** and **off** durations respectively. The on time duration Z_n and off time duration Y_n are not necessarily independent. We define $X_n \triangleq Z_n + Y_n$ and $S_n \triangleq \sum_{k=1}^{n} X_k$ for each $n \in \mathbb{N}$. It follows that $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is an *i.i.d.* sequence, and hence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a renewal sequence. We call the time interval $[S_{n-1}, S_{n-1} + Z_n]$ as nth on time followed by nth off time $(S_{n-1} + Z_n, S_n)$. We denote the distributions for on, off, and renewal periods by H, G, and F, respectively. When on and off times are independent, $F = H * G$.

We define an alternating stochastic process $W : \Omega \to \{0,1\}_+^{\mathbb{R}}$ that takes values 1 and 0, when the renewal process is in on and off state respectively. In particular, we can write alternating renewal process $W_t \triangleq \mathbb{1}_{\left\{A_t \leqslant Z_{N_t+1}\right\}}$ for any time $t \in \mathbb{R}_+$. Alternating renewal processes form an important class of renewal processes, and model many interesting applications. For each $n \in \mathbb{N}$, we observe that $W_{S_{n-1}+t} = \mathbb{1}_{[0,Z_n]}(t)$ for all $t \in [0, X_n)$. Hence, we see that the nth segment $\zeta_n = (X_n, (\mathbb{1}_{[0,Z_n]}(t) : t \in [0, X_n))$ and the segment sequence $(\zeta_n, n \in \mathbb{N})$ is *i.i.d.* . Therefore, it follows that W is a regenerative process over renewal sequence S.

Example 1.7 (Age-dependent branching process). Consider a population, where each organism $i \in \mathbb{N}$ lives for an *i.i.d.* random time period of $T_i : \Omega \to \mathbb{R}_+$ units with common distribution function $F : \mathbb{R}_+ \to [0,1]$. Just before dying, each organism produces an *i.i.d.* random number of offsprings $N : \Omega \to \mathbb{Z}_+$, with common distribution $P \in \mathcal{M}(\mathbb{N})$. Let X_t denote the number of organisms alive at time t. The stochastic process $X : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is called an **age-dependent branching process**. This is a popular model in biology for population growth of various organisms. We are interested in computing $m_t \triangleq \mathbb{E}X_t$ when $n \triangleq \mathbb{E}[N] = \sum_{j \in \mathbb{Z}_+} jP_j$.

We will show that starting from a single organism, the population including itself and its subsequent descendants is regenerative process. We take $X_0 = 1$, and let T_1 and N_1 denote the life time and number of offsprings of the first organism. If $T_1 > t$, then X_t is still equal to $X_0 = 1$. In this case, we have

$$\mathbb{E}[X_t \mathbb{1}_{\{T_1 > t\}} \mid \mathcal{F}_{T_1}] = X_0 \mathbb{1}_{\{T_1 > t\}}. \tag{1}$$

If $T_1 \leqslant t$, then $X_{T_1} = N_1$ and each of the offsprings start the population growth, independent of the past, and stochastically identical to the population growth of the original organism starting at time T_1. Hence, we can write $\mathbb{E}[X_t \mathbb{1}_{\{T_1 \leqslant t\}} \mid \mathcal{F}_{T_1}] = \mathbb{E}[\sum_{i=1}^{N_1} X_{t-T_1}^i \mathbb{1}_{\{T_1 \leqslant t\}} \mid \sigma(T_1)]$ for this case, where $(X_{T_1+u}^i, u \geqslant 0)$ is a stochastic replica of $(X_u, u \geqslant 0)$, and independent for each $i \in [N_1]$. Hence for this case, we can write the following expectation conditioned on T_1

$$\mathbb{E}[X_t \mathbb{1}_{\{T_1 \leqslant t\}} \mid \mathcal{F}_{T_1}] = \mathbb{E}[\sum_{i=1}^{N_1} X_{t-T_1}^i \mathbb{1}_{\{T_1 \leqslant t\}} \mid \sigma(T_1)] = n m_{t-T_1} \mathbb{1}_{\{T_1 \leqslant t\}}. \tag{2}$$

Example 1.8 (Renewal reward process). Consider a renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with *i.i.d.* inter renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having common distribution $F : \mathbb{R}_+ \to [0,1]$. The associated counting process is denoted by $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. We also consider an associated reward sequence $R : \Omega \to \mathbb{R}^{\mathbb{N}}$, such that a reward R_n is earned at the end of the nth renewal interval. The reward R_n can possibly depend on inter renewal time X_n, but is *i.i.d.* across intervals $n \in \mathbb{N}$. That is, we assume $(X, R) : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ be *i.i.d.* , then the **reward process** $Q : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ is defined as the accumulated reward earned by time t as

$$Q_t \triangleq \sum_{i=1}^{N_t} R_i.$$

The nth segment for process R_{N_t+1} is $\zeta_n = (X_n, R_n)$. It follows that the segment sequence $\zeta : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ is *i.i.d.* , and hence R_{N_t+1} is regenerative process with regeneration intervals being the renewal intervals $[S_{n-1}, S_n)$.