Lecture-27: Poisson process on the half-line

1 Simple point processes on the half-line

A stochastic process defined on the half-line $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a **counting process** if

1. $N_0 = 0$, and
2. for each $\omega \in \Omega$, the sample path $N(\omega) : \mathbb{R}_+ \to \mathbb{Z}_+$ is non-decreasing, integer valued, and right continuous function of time $t \in \mathbb{R}_+$.

Each discontinuity of the sample path of the counting process can be thought of as a jump of the process, as shown in Figure 1. A simple counting process has the unit jump size almost surely. General point processes in higher dimension don't have any inter-arrival time interpretation.

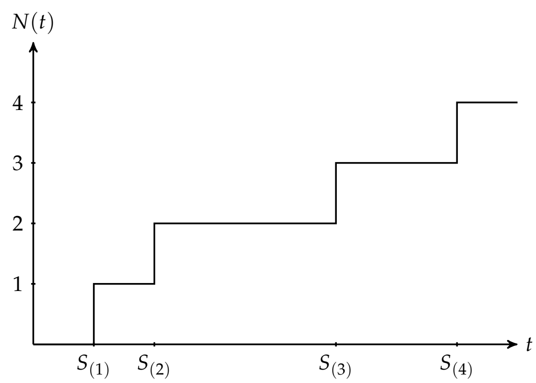

Figure 1: Sample path of a simple counting process.

Lemma 1.1. *A counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ has finitely many jumps in a finite interval $(0, t]$ almost surely.*

Definition 1.2. The points of discontinuity are also called the **arrival instants** of the counting process N. The n**th arrival instant** is a random variable denoted $\tilde{S}_n : \Omega \to \mathbb{R}_+$, defined inductively as

$$\tilde{S}_0 \triangleq 0, \qquad \tilde{S}_n \triangleq \inf\{t \geqslant 0 : N_t \geqslant n\}, \quad n \in \mathbb{N}.$$

Definition 1.3. The **inter arrival time** between $(n-1)$th and nth arrival is denoted by X_n and written as $X_n \triangleq \tilde{S}_n - \tilde{S}_{n-1}$.

Remark 1. For a simple point process, we have $P\{X_n = 0\} = P\{X_n \leqslant 0\} = 0$.

Lemma 1.4. *Simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ and arrival process $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ are inverse processes, i.e.*

$$\{\tilde{S}_n \leqslant t\} = \{N_t \geqslant n\}.$$

Proof. Let $\omega \in \{\tilde{S}_n \leqslant t\}$, then $N_{\tilde{S}_n} = n$ by definition. Since N is a non-decreasing process, we have $N_t \geqslant N_{\tilde{S}_n} = n$. Conversely, let $\omega \in \{N_t \geqslant n\}$, then it follows from definition that $\tilde{S}_n \leqslant t$. □

Corollary 1.5. *For arrival instants* $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *associated with a counting process* $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ *we have* $\{\tilde{S}_n \leqslant t, \tilde{S}_{n+1} > t\} = \{N_t = n\}$ *for all* $n \in \mathbb{Z}_+$ *and* $t \in \mathbb{R}_+$.

Proof. It is easy to see that $\{\tilde{S}_{n+1} > t\} = \{\tilde{S}_{n+1} \leqslant t\}^c = \{N_t \geqslant n+1\}^c = \{N_t < n+1\}$. Hence,

$$\{N_t = n\} = \{N_t \geqslant n, N_t < n+1\} = \{\tilde{S}_n \leqslant t, \tilde{S}_{n+1} > t\}.$$

□

Lemma 1.6. *Let* $F_n(x)$ *be the distribution function for* S_n*, then* $P_n(t) \triangleq P\{N_t = n\} = F_n(t) - F_{n+1}(t)$.

Proof. It suffices to observe that following is a union of disjoint events,

$$\{\tilde{S}_n \leqslant t\} = \{\tilde{S}_n \leqslant t, \tilde{S}_{n+1} > t\} \cup \{\tilde{S}_n \leqslant t, \tilde{S}_{n+1} \leqslant t\}.$$

□

2 IID exponential inter-arrival times characterization

Proposition 2.1. *The counting process* $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ *associated with a simple Poisson point process* $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *is Markov.*

Proof. We define the event space $\mathcal{F}_t \triangleq \sigma(N_s : s \leqslant t)$ as the history of the process until time $t \in \mathbb{R}_+$. Then, from the independent increment property of Poisson processes, we have for any historical event $H_s \in \mathcal{F}_s$

$$P(\{N_t = n\} \mid H_s \cap \{N_s = k\}) = P(\{N_t - N_s = n - k\} \mid H_s \cap \{N_s = k\}) = P(\{N_t = n\} \mid \{N_s = k\}).$$

For a homogeneous Poisson point process, the process is homogeneously Markov with $P(\{N_t = n\} \mid \{N_s = k\}) = P\{N(t-s) = n-k\} = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{n-k}}{(n-k)!}$. □

Theorem 2.2. *The counting process* $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ *associated with a simple Poisson point process* $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *is strongly Markov.*

Proposition 2.3. *A simple counting process* $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ *is a* ***homogeneous Poisson process*** *with a finite positive rate* λ*, iff the inter-arrival time sequence* $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *are* i.i.d. *random variables with an exponential distribution of rate* λ.

Proof. We first assume the *i.i.d.* exponentially distributed inter-arrival times to show that for the simple counting process N and each positive real $t \in \mathbb{R}_+$, the random variable N_t is Poisson with parameter λt, and hence N is homogeneous Poisson with rate λ from the equivalence ii_- in Theorem **??**.

For the converse, let N_t be a simple homogeneous Poisson point process on half-line with rate λ. From equivalence iii_- in Theorem **??**, we obtain for any positive integer t,

$$P\{X_1 > t\} = P\{N_t = 0\} = e^{-\lambda t}.$$

It suffices to show that inter-arrivals time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* . We can show that N is Markov process with strong Markov property. Since the sequence of ordered points $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a sequence of stopping times for the counting process, it follows from the strong Markov property of this process that $(N_{\tilde{S}_n+t} - N_{\tilde{S}_n} : t \geqslant 0)$ is independent of $\sigma(N_s : s \leqslant \tilde{S}_n)$ and hence of $\tilde{S}_n$ and $N_{\tilde{S}_n}$. Further, we see that

$$X_{n+1} = \inf\left\{t > 0 : N_{\tilde{S}_n+t} - N_{\tilde{S}_n} = 1\right\}.$$

It follows that $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is an independent sequence. For homogeneous Poisson point process, we have $N_{\tilde{S}_n+t} - N_{\tilde{S}_n} = N_t$ in distribution, and hence X_{n+1} has same distribution as X_1 for each $n \in \mathbb{N}$.

□

For many proofs regarding Poisson processes, we partition the sample space with the disjoint events $\{N_t = n\}$ for $n \in \mathbb{Z}_+$. We need the following lemma that enables us to do that.

Lemma 2.4. *For any finite time $t > 0$, a Poisson process is finite almost surely.*

Proof. By strong law of large numbers, we have

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X_1] = \frac{1}{\lambda} \quad \text{a.s.}$$

Fix $t > 0$ and we define a sample space subset $M = \{\omega \in \Omega : N(\omega, t) = \infty\}$. For any $\omega \in M$, we have $S_n(\omega) \leqslant t$ for all $n \in \mathbb{N}$. This implies $\limsup_n \frac{S_n}{n} = 0$ and $\omega \notin \left\{\lim_n \frac{S_n}{n} = \frac{1}{\lambda}\right\}$. Hence, the probability measure for set M is zero. □

2.1 Distribution functions

Lemma 2.5. *Moment generating function of arrival times $\tilde{S}_n$ is*

$$M_{\tilde{S}_n}(t) = \mathbb{E}[e^{t\tilde{S}_n}] = \begin{cases} \frac{\lambda^n}{(\lambda-t)^n}, & t < \lambda \\ \infty, & t \geqslant \lambda. \end{cases}$$

Lemma 2.6. *Distribution function of $\tilde{S}_n$ is given by $F_n(t) \triangleq P\left\{\tilde{S}_n \leqslant t\right\} = 1 - e^{-\lambda t}\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}$.*

Theorem 2.7. *Density function of $\tilde{S}_n$ is Gamma distributed with parameters n and λ. That is,*

$$f_n(s) = \frac{\lambda(\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}.$$

Theorem 2.8. *For each $t > 0$, the distribution of Poisson process N_t with parameter λ is given by*

$$P\{N_t = n)\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

Further, $\mathbb{E}[N_t] = \lambda t$, explaining the rate parameter λ for Poisson process.

Proof. Result follows from density of $\tilde{S}_n$ and recognizing that $P_n(t) = F_n(t) - F_{n+1}(t)$. □

Corollary 2.9. *Distribution of arrival times $\tilde{S}_n$ is*

$$F_n(t) = \sum_{j\geqslant n} P_j(t), \qquad \sum_{n\in\mathbb{N}} F_n(t) = \mathbb{E}N_t.$$

Proof. First result follows from the telescopic sum and the second from the following observation.

$$\sum_{n\in\mathbb{N}} F_n(t) = \mathbb{E}\sum_{n\in\mathbb{N}} 1\{N_t \geqslant n\} = \sum_{n\in\mathbb{N}} P\{N_t \geqslant n\} = \mathbb{E}N_t.$$

□

A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant. This is clear from looking at the first moment $\mathbb{E}N_t = \lambda t$, which is linearly increasing in time.