Lecture-09: Limit Theorems

1 Growth of renewal counting processes

Lemma 1.1. *Consider the counting process* $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ *associated with* i.i.d. *inter-renewal time sequence* $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *with finite mean* $\mathbb{E}X_n < \infty$*. Let* $N_\infty \triangleq \lim_{t\to\infty} N_t$*, then* $P\{N_\infty = \infty\} = 1$.

Proof. It suffices to show $P\{N_\infty < \infty\} = 0$. Since $\mathbb{E}[X_n] < \infty$, we have $P\{X_n = \infty\} = 0$ and

$$P\{N_\infty < \infty\} = P\bigcup_{n\in\mathbb{N}}\{N_\infty < n\} = P\bigcup_{n\in\mathbb{N}}\{S_n = \infty\} = P\bigcup_{n\in\mathbb{N}}\{X_n = \infty\} \leqslant \sum_{n\in\mathbb{N}} P\{X_n = \infty\} = 0.$$

□

Corollary 1.2. *For delayed renewal processes with finite mean of first renewal instant and subsequent inter-renewal times,* $P\left\{\lim_{t\to\infty} N_t^D = \infty\right\} = 1$.

1.1 Strong law for renewal processes

We observed that the number of renewals N_t increases to infinity with the length of the duration t. We will show that the growth of N_t is asymptotically linear with time t, and we will find this coefficient of linear growth of N_t with time t.

Theorem 1.3 (Strong law). *For a renewal counting process* $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ *with* i.i.d. *inter-renewal times* $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *having a finite mean* μ*, we have*

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu} \text{ almost surely.}$$

Proof. Note that S_{N_t} represents the time of last renewal before t, and S_{N_t+1} represents the time of first renewal after time t. Clearly, we have $S_{N_t} \leqslant t < S_{N_t+1}$. Dividing by N_t, we get

$$\frac{S_{N_t}}{N_t} \leqslant \frac{t}{N_t} < \frac{S_{N_t+1}}{N_t}. \quad (1)$$

Since N_t increases monotonically to infinity as t grows large, we can apply strong law of large numbers to the sum $S_{N_t} = \sum_{i=1}^{N_t} X_i$, to get $\lim_{t\to\infty} \frac{S_{N_t}}{N_t} = \mu$ almost surely. Hence the result follows. □

Corollary 1.4. *For a delayed renewal process with finite inter-arrival durations,* $\lim_{t\to\infty} \frac{N_D(t)}{t} = \frac{1}{\mu_F}$.

Example 1.5. Suppose, you are in a casino with infinitely many games. We assume that $X : \Omega \to [0,1]^{\mathbb{N}}$ is an *i.i.d.* uniform sequence where X_i is the random probability of win in the game $i \in \mathbb{N}$. One can continue to play a game or switch to another one. We are interested in a strategy that maximizes the long-run proportion of wins. Let $N(n)$ denote the number of losses in n plays. Then the fraction of wins $P_W(n)$ is given by $P_W(n) = \frac{n-N(n)}{n}$.

We pick a strategy where any game is selected to play, and continue to be played till the first loss. We show that $\lim_{n\to\infty} P_W(n) = 1$ for this proposed strategy. Let T_i be the number of times a game i is played. We observe that the conditional probability mass function for the number of plays for each game i is geometrically distributed as

$$\mathbb{E}[\mathbb{1}_{\{T_i=k\}} \mid \sigma(X_i)] = X_i^{k-1}(1 - X_i), \quad k \in \mathbb{N}.$$

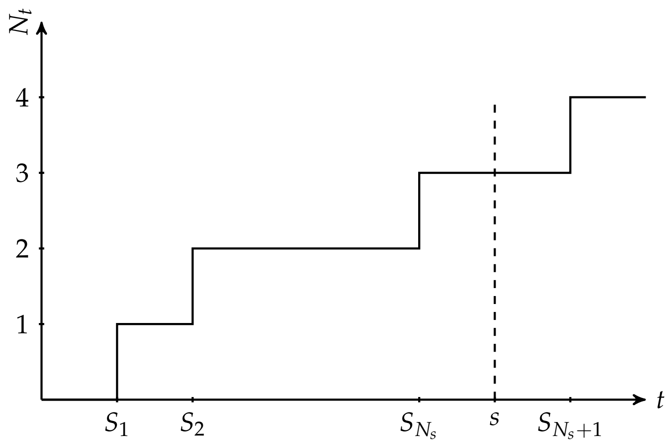

Figure 1: Time of last renewal

Hence, it follows that T_i are *i.i.d.* random variables with mean $\mathbb{E}T_i = \mathbb{E}[\mathbb{E}[T_i \mid X_i]] = \mathbb{E}\left[\frac{1}{1-X_i}\right] = \infty$. It follows that each loss is a renewal event, and from the strong law of renewal process, we obtain

$$\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mathbb{E}[\text{Time till first loss}]} = \frac{1}{\mathbb{E}T_i} = 0.$$

1.2 Wald's lemma for renewal processes

Basic renewal theorem implies $\frac{N_t}{t}$ converges to $\frac{1}{\mu}$ almost surely. We are next interested in convergence of the ratio $\frac{m_t}{t}$. Note that this is not obvious, since almost sure convergence doesn't imply convergence in mean. To illustrate this, we have the following example.

Example 1.6. Consider a Bernoulli random sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ with probability $P\{X_n = 1\} = \frac{1}{n}$, and another random sequence $Y : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ defined as $Y_n \triangleq nX_n$ for $n \in \mathbb{N}$. Then, $P\{Y_n = 0\} = 1 - \frac{1}{n}$. That is $Y_n \to 0$ a.s. However, $\mathbb{E}[Y_n] = 1$ for all $n \in \mathbb{N}$. So $\mathbb{E}[Y_n] \to 1$.

Even though, basic renewal theorem does **NOT** imply it, we still have $\frac{m_t}{t}$ converging to $\frac{1}{\mu}$. We first need this technical Lemma.

Proposition 1.7 (Wald's Lemma for renewal process). *Let $m : \mathbb{R}_+ \to \mathbb{R}_+$ be the renewal function for a renewal counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with* i.i.d. *inter-arrival times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having finite mean $\mu = \mathbb{E}[X_1] < \infty$. Then, $N_t + 1$ is a stopping time for the sequence X, and*

$$\mathbb{E}\left[\sum_{i=1}^{N_t+1} X_i\right] = \mu(1 + m_t).$$

Proof. We observe that for any $n \in \mathbb{N}$, the event $\{N_t + 1 = n\}$ belongs to $\sigma(X_1, \dots, X_n)$, since

$$\{N_t + 1 = n\} = \{S_{n-1} \leqslant t < S_n\} = \left\{\sum_{i=1}^{n-1} X_i \leqslant t < \sum_{i=1}^{n-1} X_i + X_n\right\} \in \sigma(X_1, \dots, X_n).$$

Thus $N_t + 1$ is a stopping time with respect to the random sequence X, and the result follows from Wald's Lemma. □

Theorem 1.8 (Elementary renewal theorem). *For a renewal process with finite mean inter-arrival times, the renewal function satisfies*

$$\lim_{t\to\infty} \frac{m_t}{t} = \frac{1}{\mu}.$$

Proof. By the assumption, we have mean $\mu < \infty$. Further, we know that $S_{N_t+1} > t$. Taking expectations on both sides and using Proposition 1.7, we have $\mu(m_t + 1) > t$. Dividing both sides by μt and taking lim inf on both sides, we get

$$\liminf_{t\to\infty} \frac{m_t}{t} \geqslant \frac{1}{\mu}.$$

We employ a truncated random variable argument to show the reverse inequality. We define truncated inter-renewal times $\bar{X} : \Omega \to [0, M]^{\mathbb{N}}$ defined as $\bar{X}_n \triangleq X_n \wedge M$ for each $n \in \mathbb{N}$, and with common mean denoted by μ_M. Since X is *i.i.d.* , so is the truncated sequence $\bar{X}$, and hence we can define the corresponding renewal sequence $\bar{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and the counting process $\bar{N} : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ defined as

$$\bar{S}_n \triangleq \sum_{i=1}^{n} \bar{X}_i, \quad n \in \mathbb{N}, \qquad \text{and } \bar{N}_t \triangleq \sum_{n\in\mathbb{N}} 1_{\{\bar{S}_n \leqslant t\}}, \quad t \in \mathbb{R}_+.$$

Note that since $S_n \geqslant \bar{S}_n$, the number of arrivals would be higher for renewal process $\bar{N}_t$ with truncated random variables. That is, $N_t \leqslant \bar{N}_t$, and hence $m_t \leqslant \bar{m}_t$ from the monotonicity of expectation. Further, due to truncation of inter-arrival time, next renewal happens within M units of time, that is $\bar{S}_{\bar{N}_t+1} \leqslant t + M$. From the monotonicity of expectation and Wald's Lemma for renewal processes, we get

$$(1 + \bar{m}_t)\mu_M \leqslant t + M.$$

Dividing both sides by $t\mu_M$ and the fact that $m_t \leqslant \bar{m}_t$ for all times $t \in \mathbb{R}_+$, we obtain

$$\limsup_{t\to\infty} \frac{m_t}{t} \leqslant \limsup_{t\to\infty} \frac{\bar{m}_t}{t} \leqslant \frac{1}{\mu_M}.$$

The result follows from recognizing that $\lim_{M\to\infty} \mu_M = \mu$. □

Corollary 1.9. *For a delayed renewal process with finite inter-arrival durations, we have* $\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mu_F}$.

Example 1.10 (Markov chain). Consider a positive recurrent discrete time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ taking values in a discrete set $\mathcal{X} \subset \mathbb{R}$. Let the initial state be $X_0 = x \in \mathcal{X}$ and $\tau_y^+(0) = 0$ for $y \neq x \in \mathcal{X}$, then we can inductively define the nth recurrent time to state y as a stopping time

$$\tau_y^+(n) = \inf\left\{k > \tau_y^+(n-1) : X_k = y\right\}.$$

Since any discrete time Markov chain satisfies the strong Markov property, it follows that $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ form a delayed renewal process with the first arrival distribution $P_x\left\{\tau_y^+(1) = k\right\} = f_{xy}^{(k)}$, and the common distribution of the inter-arrival duration $X_n, n \geqslant 2$ in terms of first return probability as

$$P_y\left\{\tau_y^+(1) = k\right\} = f_{yy}^{(k)},\ k \in \mathbb{N}.$$

We denote the associated counting process by $N_y : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$, where $N_y(n) = \sum_{i\in\mathbb{N}} \mathbb{1}_{\{\tau_y^+(i) \leqslant n\}} = \sum_{k=1}^{n} \mathbb{1}_{\{X_k = y\}}$ denotes the number of visits to state y up to time n. Let $\mu_{yy} = \mathbb{E}_y \tau_y^+(1)$ be the finite mean inter-arrival time for the renewal process, also the mean recurrence time to state y. From the strong law for delayed renewal processes it follows that

$$P_y\left\{\lim_{n\in\mathbb{N}} \frac{N_y(n)}{n} = \frac{1}{\mu_{yy}}\right\} = 1.$$

Since $N_y(n)$ is number of visits to state y in first n time steps, we have $\mathbb{E}_x N_y(n) = \sum_{k=1}^{n} P_x\{X_k = y\} = \sum_{k=1}^{n} p_{xy}^{(k)}$ From the basic renewal theorem for delayed renewal process it follows that

$$\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = \lim_{n\in\mathbb{N}} \frac{\mathbb{E}_x[N_y(n)]}{n} = \frac{1}{\mu_{yy}}.$$

1.3 Central limit theorem for renewal processes

Theorem 1.11. *For a renewal process with inter-arrival times having finite mean μ and finite variance σ^2, the associated counting process converges to a normal random variable in distribution. Specifically,*

$$\lim_{t\to\infty} P\left\{\frac{N_t - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} < y\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}dx.$$

Proof. Take $u = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$. We shall treat u as an integer and proceed, the proof for general u is an exercise. Recall that $\{N_t < u\} = \{S_u > t\}$. By equating probability measures on both sides, we get

$$P\{N_t < u\} = P\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > \frac{t-u\mu}{\sigma\sqrt{u}}\right\} = P\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > -y\left(1+\frac{y\sigma}{\sqrt{t\mu}}\right)^{-1/2}\right\}.$$

By central limit theorem, $\frac{S_u-u\mu}{\sigma\sqrt{u}}$ converges to a normal random variable with zero mean and unit variance as t grows. We also observe that

$$\lim_{t\to\infty} -y\left(1+\frac{y\sigma}{\sqrt{tu}}\right)^{-1/2} = -y.$$

These results combine with the symmetry of normal random variable to give us the result. □

2 Patterns

Let $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ be an *i.i.d.* sequence with common probability mass function $p \in \mathcal{M}(\mathcal{X})$. We denote the natural filtration of process X by $\mathcal{F}_\bullet \triangleq (\mathcal{F}_n : n \in \mathbb{N})$ where $\mathcal{F}_n \triangleq \sigma(X_1,\dots,X_n)$ for all $n \in \mathbb{N}$. Let $x = (x_1,\dots,x_m) \in \mathcal{X}^m$ be a pattern and inductively define nth hitting times of the pattern x as $S_0^x \triangleq 0$ and

$$S_n^x \triangleq \inf\left\{n > S_{n-1}^x : X_n = x_m, X_{n-1} = x_{m-1},\dots,X_{n-m+1} = x_1\right\}.$$

It is easy to check that S_n^x is adapted to $\mathcal{F}_\bullet$ and one can verify that S_n^x is almost surely finite for all $n \in \mathbb{N}$. It follows that S^x is a sequence of stopping times adapted to $\mathcal{F}_\bullet$. Since X is *i.i.d.* , it follows that $S^x : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a delayed renewal sequence with inter-renewal durations $T_n^x \triangleq S_n^x - S_{n-1}^x$ being *i.i.d.* for $n \geqslant 2$ and independent of T_1^x.

2.1 Hitting time to pattern (1)

First we consider the simplest example when the alphabet $\mathcal{X} = \{0,1\}$, with the common mean $\mathbb{E}X_1 = p$, and the pattern $x = (1)$. One way to solve this problem is to consider S_1^1 as a random variable and find its distribution. We can write

$$P\left\{S_1^1 = k\right\} = \bar{p}^{k-1}p.$$

We observe that S_1^1 is a geometric random variable of the time to first success, with its mean as the reciprocal of *i.i.d.* success probability p. An alternative way to solve this is via renewal function approach. Recall that $\left\{S_1^1 = 1\right\} = \{X_1 = 1\}$ and $S_1^1\mathbb{1}_{\{X_1=0\}} = (1+S_1^1)\mathbb{1}_{\{X_0=0\}}$ in distribution where S_1^1 is independent of X_0. The result follows from writing

$$\mathbb{E}S_1^1 = \mathbb{E}S_1^1\mathbb{1}_{\{S_1^1>1\}} + \mathbb{E}S_1^1\mathbb{1}_{\{S_1^1=1\}} = \bar{p}\mathbb{E}(1+S_1^1) + p = 1 + \bar{p}\mathbb{E}S_1^1.$$

2.2 Hitting time to pattern $(0,1)$

For the alphabet $\mathcal{X} = \{0,1\}$ with common mean $\mathbb{E}X_1 = p$, we consider the two length pattern $x = (0,1)$, then $S_1^x = \inf\{n \in \mathbb{N} : X_n = 1, X_{n-1} = 0\}$. We can again model this hitting time as a random variable, however directly finding the distribution of S^x is slightly more complicated. We next attempt the renewal function approach. Recall that S_1^x is independent of X_0, and the following equality holds in distribution $S_1^x\mathbb{1}_{\{X_1=1\}} = (1+S_1^x)\mathbb{1}_{\{X_0=1\}}$. In addition, the following equality holds in distribution $S_1^x\mathbb{1}_{\{X_1=0,X_2=0\}} = (1+S_1^x)\mathbb{1}_{\{X_1=0,X_0=0\}}$. Hence, we can write

$$\mathbb{E}S_1^x = \mathbb{E}S_1^x\mathbb{1}_{\{X_1=0\}} + \mathbb{E}S_1^x\mathbb{1}_{\{X_1=1\}} = \mathbb{E}S_1^x\mathbb{1}_{\{X_2=1,X_1=0\}} + \mathbb{E}S_1^x\mathbb{1}_{\{X_2=0,X_1=0\}} + p\mathbb{E}(1+S_1^x).$$

We recognize that the second term on the right hand side can be written as

$$\mathbb{E}S_1^x \mathbb{1}_{\{X_2=0,X_1=0\}} = \bar{p}\mathbb{E}(1+S_1^x)\mathbb{1}_{\{X_1=0\}} = \bar{p}^2 + \bar{p}\mathbb{E}S_1^x\mathbb{1}_{\{X_1=0\}} = \bar{p}^2 + \bar{p}\mathbb{E}S_1^x - \bar{p}p\mathbb{E}(1+S_1^x).$$

Combining the above two results, we can write

$$\mathbb{E}S_1^x = 2p\bar{p} + \bar{p}^2 + \bar{p}\mathbb{E}S_1^x + p^2\mathbb{E}(1+S_1^x) = 1 + (\bar{p}+p^2)\mathbb{E}S_1^x.$$

It follows that $\mathbb{E}S_1^x = \frac{1}{p\bar{p}}$.

2.3 Hitting time to pattern x

For an *i.i.d.* sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, a general approach is to model $X_n^m = (X_n, X_{n-1}, X_{n-m+1}) \in \mathcal{X}^m$ as an m-dimensional time homogeneous irreducible positive recurrent Markov chain. Defining $\mathcal{Y} \triangleq \mathcal{X}^m$ We are interested in the mean hitting time to state x of the joint process $X^m : \Omega \to \mathcal{Y}^{\mathbb{N}}$. It follows that the successive times for process X^m to hit a pattern $x \in \mathcal{Y}$ is a delayed renewal process in general. Defining the on times when X^m hits x, it follows from the strong law for renewal processes that the average number of visits to state x is the reciprocal of mean inter-renewal duration. That is,

$$\lim_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{\{X_n^m = x\}} = \lim_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{\{X_n = x_m, \dots, X_{n-m+1} = x_1\}} = \prod_{i=1}^{m} p_{x_i} = \frac{1}{\mathbb{E}T_k^x}.$$

For each pattern $x \in \mathcal{Y}$, we define initial sub-patterns $x^k \triangleq (x_1, \dots, x_k)$ for $k \in [m]$. If the initial sub-pattern is not one of the final sub-patterns, i.e. $(x_1, \dots, x_k) \neq (x_{m-k+1}, \dots, x_m)$ for any $k \in [m]$, then we observe that S^x is a renewal sequence and $\mathbb{E}S_1^x = \frac{1}{\prod_{i=1}^m p_{x_i}}$.

Example 2.1. Consider patterns (1) and (01) for *i.i.d.* Bernoulli sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ with common mean $\mathbb{E}X_1 = p$. Clearly, (1) has no sub-pattern and hence $\mathbb{E}S_1^1 = \frac{1}{p}$. Similarly, (01) has a sub-pattern (0) but (00) is not a sub-pattern of (01) and hence $\mathbb{E}S_1^{01} = \frac{1}{\bar{p}p}$.

If there exists a non empty $I \subseteq [m]$ such that for each $k \in I$ there is an initial sub-pattern x^k such that $(x^k) = (x^m_{m-k+1})$ is a final sub-pattern of x, then the mean hitting time to pattern x is equal to the telescopic sum of mean hitting time to sub-patterns That is, denoting $I \triangleq \{i_1, \dots, i_k\}$, we can write

$$\mathbb{E}S_1^x = \sum_{j=1}^{k} \mathbb{E}_{x^{i_j}} S_1^{x^{i_{j+1}}}.$$

The mean time duration between two successive hits to $x^{i_{j+1}}$ is $\mathbb{E}T_2^{x^{i_{j+1}}} = \frac{1}{\prod_{\ell=1}^{i_{j+1}} p_{x_\ell}}$. This is the same mean time from x^{i_j} to $x^{i_{j+1}}$. Therefore,

$$\mathbb{E}S_1^x = \sum_{j=1}^{k} \frac{1}{\prod_{\ell=1}^{i_{j+1}} p_{x_\ell}}.$$

Example 2.2. Consider pattern (101) and (1011) for *i.i.d.* Bernoulli sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ with common mean $\mathbb{E}X_1 = p$. Pattern (101) has sub-patterns (1) and (10), where (1) appears at the end as well. Therefore,

$$\mathbb{E}S_1^{(101)} = \mathbb{E}S_1^1 + \mathbb{E}_1 S_1^{101} = \frac{1}{p} + \frac{1}{p^2\bar{p}}.$$

Pattern (1011) has sub-patterns $(1), (10), (101)$, where (1) appears at the end. Thus, Therefore,

$$\mathbb{E}S_1^{(101)} = \mathbb{E}S_1^1 + \mathbb{E}_1 S_1^{1011} = \frac{1}{p} + \frac{1}{p^3\bar{p}}.$$