Lecture-10: Regenerative Processes

1 Regenerative processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be a renewal sequence, with the associated inter renewal sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. That is, the nth renewal instant is $S_n \triangleq \sum_{i=1}^n X_i$ for each $n \in \mathbb{N}$ and the number of renewals is $N_t \triangleq \sum_{n \in \mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$ until each time $t \in \mathbb{R}_+$.

Definition 1.1. Consider a stochastic process $Z : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ defined over the same probability space. The nth segment of the joint process $(N, Z) : \Omega \to (\mathbb{Z}_+ \times \mathbb{R})^{\mathbb{R}_+}$ is defined as the sample path in the nth inter renewal duration, written $\zeta_n \triangleq (X_n, (Z_{S_{n-1}+t} : t \in [0, X_n))$ for each $n \in \mathbb{N}$.

Definition 1.2. The process Z is *regenerative* over the renewal sequence S, if its segments $(\zeta_n : n \in \mathbb{N})$ are *i.i.d.* . The process Z is *delayed regenerative*, if S is a delayed renewal sequence and the segments $(\zeta_n : n \in \mathbb{N})$ of the joint process are independent with $(\zeta_n : n \geqslant 2)$ being identically distributed.

Definition 1.3. Let $\mathcal{F}_t \triangleq \sigma(N_u, Z_u, u \leqslant t)$ be the history of the regenerative process until time $t \in \mathbb{R}_+$. The renewal sequence S is the *regeneration times* for the process Z, and the process Z possesses the *regenerative property* of the process $(Z_{S_{n-1}+t} : t \geqslant 0)$ being independent of history $\mathcal{F}_{S_{n-1}}$ and distributed identically to Z. For delayed regenerating process Z, the process $(Z_{S_{n-1}+t} : t \geqslant 0)$ is independent of history $\mathcal{F}_{S_{n-1}}$ and is identically distributed for $n \geqslant 2$.

Remark 1. The definition says that probability law is independent of the past and shift invariant at renewal times. That is after each renewal instant, the process becomes an independent probabilistic replica of the process starting from zero for the regenerative process. The process starting at each renewal instant is an independent probabilistic replica of the process starting from the first renewal instant for the delayed regenerative process.

Remark 2. If the regenerative process Z is bounded, then for any Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}[f(Z_t) \mid \mathcal{F}_{S_{n-1}}] = \mathbb{E}[f(Z_{t-S_{n-1}}) \mid \sigma(S_{n-1})]\mathbb{1}_{\{t \geqslant S_{n-1}\}} + f(Z_t)\mathbb{1}_{\{t < S_{n-1}\}}.$$

Example 1.4 (Markov chains). Consider a discrete time homogeneous, irreducible, aperiodic and finite state Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ and a state $y \in \mathcal{X}$. Then X is positive recurrent and we denote its invariant distribution by $\pi \in \mathcal{M}(\mathcal{X})$. We can inductively define the recurrent times for state y as $\tau_y^+(0) \triangleq 0$ and $\tau_y^+(k) \triangleq \inf\left\{n > \tau_y^+(k-1) : X_n = y\right\}$. For each $k \in \mathbb{N}$, kth recurrent time to state y is adapted to natural filtration of Markov chain X and is almost surely finite since X is positive recurrent. It follows that $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ is a sequence of stopping times.

We fix $k \in \mathbb{N}$, and define the kth excursion time to the state y as $I_k \triangleq \{\tau_y^+(k-1) + 1, \dots, \tau_y^+(k)\}$ and the length of this excursion as $T_y(k) \triangleq \tau_y^+(k) - \tau_y^+(k-1)$. We can write the kth segment for the Markov chain X as

$$\zeta_k \triangleq (T_y(k), (X_{\tau_y^+(k-1)+m} : m \in [T_y(k)]).$$

From the strong Markov property of Markov chain X applied to each stopping time $\tau_y^+(k)$, we observe that $(X_{\tau_y^+(k)+n} : n \in \mathbb{Z}_+)$ is independent of the random past $\mathcal{F}_{\tau_y^+(k)}$ and distributed identically for $k \in \mathbb{N}$. It follows that the segments are independent and distributed identically for $k \geqslant 2$. We can write the joint distribution for $(T_y(k), X_{\tau_y^+(k-1)+m})$ for $m < T_y(k), z \neq y$, and $k \geqslant 2$, as

$$P\left\{m < T_y(k) = r, X_{\tau_y^+(k-1)+m} = z\right\} = P_y\left\{\tau_y^+(1) > m, X_m = z\right\} P_z\left\{\tau_y^+(1) = r - m\right\}.$$

The equality follows from the strong Markov property and the homogeneity of process X. It follows that the Markov process X is a delayed regenerative process over delayed renewal sequence τ_y^+.

1.1 Renewal equation

Let $Z:\Omega\to\mathcal{Z}^{\mathbb{R}_+}$ be a regenerative process over renewal sequence $S:\Omega\to\mathbb{R}_+^{\mathbb{N}}$ defined on the probability space $(\Omega,\mathcal{F},P)$, and F be the distribution of inter renewal times. The counting process associated with the renewal sequence S is denoted by N, and we define the history of the joint process N,Z until time t by $\mathcal{F}_t$. We next compute the marginal distribution of a regenerative process sampled at any time t.

Definition 1.5. For a regenerative process Z and a Borel measurable set $A\in\mathcal{B}(\mathcal{Z})$, we define the marginal probability function $f\in[0,1]^{\mathbb{R}_+}$ and the kernel function $K\in[0,1]^{\mathbb{R}_+}$ at any time $t\in\mathbb{R}_+$, as

$$f_t \triangleq P\{Z_t\in A\}, \qquad K_t\triangleq P\{S_1>t, Z_t\in A\}.$$

Remark 3. Computing the marginal distribution of Z_t is equivalent to computing f_t for any measurable set $A\in\mathcal{B}(\mathcal{Z})$. We will write the marginal distribution in terms of the inter renewal time distribution F, and the kernel function K which is typically easy to compute for any regenerative process.

Proposition 1.6 (Renewal equation). *The marginal distribution f of regenerative process Z satisfies the following fixed point equation referred to as* renewal equation *in terms of associated kernel function K and inter renewal time distribution F,*

$$f = K + F * f. \tag{1}$$

Proof. We can write the probability of the event $\{Z_t\in A\}$ by partitioning it into disjoint events as

$$P\{Z_t\in A\} = P\{Z_t\in A, S_1>t\} + P\{Z_t\in A, S_1\leqslant t\}. \tag{2}$$

By the regeneration property applied at renewal instant S_1, we have

$$\mathbb{E}[\mathbb{1}_{\{Z_t\in A, S_1\leqslant t\}}\mid \mathcal{F}_{S_1}] = \mathbb{E}[\mathbb{1}_{\{Z_{t-S_1}\in A\}}\mid \sigma(S_1)]\mathbb{1}_{\{S_1\leqslant t\}} = f_{t-S_1}\mathbb{1}_{\{S_1\leqslant t\}}. \tag{3}$$

Taking expectation of (3) and combining with (2), we obtain the result. □

Theorem 1.7 (Solution to renewal equation). *The renewal equation* (1) *has a unique solution $f=(1+m)*K$, where $m=\sum_{n\in\mathbb{N}}F_n$ is the renewal function associated with the inter renewal time distribution F.*

Proof. We first verify that $f=(1+m)*K$ satisfies the renewal equation. This follows from the fact that $F*(1+m)=m$. We next show the uniqueness of this solution, by showing that if there is any other solution g to the renewal equation, then $h\triangleq g-(1+m)*K=0$. We observe that for any solution g to the renewal equation, we have $g=K+F*g$, and the difference $h=g-(1+m)*K$ satisfies $h=F*h$. By repeated application of this equation, we obtain that $h=F_n*h$ for all $n\in\mathbb{N}$. From the finiteness of m_t, it follows that $(F_n)_t\to 0$ as n grows. Hence, $\lim_{n\in\mathbb{N}}(F_n*h)_t=0$ for each $t\in\mathbb{R}_+$. □

Example 1.8 (Age and excess time processes). Let $N:\Omega\to\mathbb{R}_+^{\mathbb{R}_+}$ be the renewal counting process for the renewal sequence $S:\Omega\to\mathbb{R}_+^{\mathbb{N}}$. We define the corresponding age process $A:\Omega\to\mathbb{R}_+^{\mathbb{R}_+}$ and the excess time process $Y:\Omega\to\mathbb{R}_+^{\mathbb{R}_+}$ at each time $t\in\mathbb{R}_+$ in the following. The *age of last renewal* at time t is defined as $A_t\triangleq t-S_{N_t}$ and the *excess time for next renewal* at time t is defined as $Y_t\triangleq S_{N_t+1}-t$. Let $t\in[0,X_n)$, then $S_{n-1}+t$ corresponds to a time instant in the nth renewal interval, and we observe that the sample path of age and excess time in this renewal interval are given by

$$A_{S_{n-1}+t}=t, \qquad Y_{S_{n-1}+t}=X_n-t.$$

We can respectively write the nth segment of age and excess time processes as

$$\zeta_n\triangleq(X_n,(t:t\in[0,X_n))), \qquad \eta_n\triangleq(X_n,(X_n-t:t\in[0,X_n))).$$

Since inter renewal times are *i.i.d.* , it follows that $((\zeta_n,\eta_n):n\in\mathbb{N})$ is an *i.i.d.* sequence, and hence the age process A and excess times process Y are regenerative. However, we note that the nth segment of age and excess times are dependent. We can write the respective kernel functions K^A, K^Y for age and excess time processes, in terms of the complementary distribution function $\bar{F}$ of the inter renewal times, as

$$K_t^A\triangleq P\{A_t\geqslant x, S_1>t\}=\mathbb{1}_{\{t\geqslant x\}}\bar{F}_t, \qquad K_t^Y\triangleq P\{Y_t\geqslant x, S_1>t\}=\bar{F}_{t+x}.$$

Since the age and excess time processes are regenerative, we can apply Theorem 1.7 to compute the marginal distributions of these processes sampled at any time $t \in \mathbb{R}_+$, as

$$P\{A_t \geqslant x\} = \mathbb{1}_{\{t \geqslant x\}}\bar{F}_t + \int_0^t dm_y \mathbb{1}_{\{t-y \geqslant x\}}\bar{F}_{t-y}, \qquad P\{Y_t \geqslant x\} = \bar{F}_{t+x} + \int_0^t dm_y \bar{F}_{t+x-y}.$$

1.2 Delayed renewal equation

Let $Z : \Omega \to \mathcal{Z}^{\mathbb{R}_+}$ be a delayed regenerative process over delayed renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ defined on the probability space $(\Omega, \mathcal{F}, P)$. We assume that inter-renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is independent with distributions $G \triangleq F_{X_1}$ and $F \triangleq F_{X_n}$ for all $n \geqslant 2$. The counting process associated with the delayed renewal sequence S is denoted by N^D, and we define the history of the joint process N^D, Z until time t by $\mathcal{F}_t$.

Definition 1.9. We denote the distribution of nth renewal instant S_n by $G * F_{n-1}$ where F_n is n-fold convolution of F, and the delayed renewal function by $m^D = \sum_{n \in \mathbb{N}} G * F_{n-1}$. The nth segment of the joint process (N^D, Z) is given by $\zeta_n \triangleq (X_n, (Z_t, t \in [S_{n-1}, S_n) : n \in \mathbb{N})$.

Remark 4. For the delayed regenerative process, the segments ζ are independent and $(\zeta_n : n \geqslant 2)$ is *i.i.d.*. In this case, $(Z_{S_n+t} : t \in [0, X_{n+1}))$ is independent of $\mathcal{F}_{S_n}$ and distributed identically to $(Z_{S_1+t}, t \in [0, X_2))$ for all $n \in \mathbb{N}$.

Definition 1.10. For a delayed regenerative process Z and a Borel measurable set $A \in \mathcal{B}(\mathcal{Z})$, we define the marginal probability function $f \in [0,1]^{\mathbb{R}_+}$ and kernel functions $K^1, K^2 \in [0,1]^{\mathbb{R}_+}$ at any time $t \in \mathbb{R}_+$,

$$f_t \triangleq P\{Z_t \in A\}, \qquad K_t^1 \triangleq P\{S_1 > t, Z_t \in A\}, \qquad K_t^2 \triangleq P\{Z_{S_1+t} \in A, t \in [0, X_2)\}.$$

Theorem 1.11. *The marginal distribution f of delayed regenerative process Z can be written in terms of associated kernel functions K^1, K^2 and delayed renewal function m^D, as*

$$f = K^1 + K^2 * m^D. \tag{4}$$

Proof. Fix a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$. We can write the probability $f_t = P\{Z_t \in A\}$ as the sum of probability of disjoint partitions $(\{Z_t \in A, N_t = n\} : n \in \mathbb{N})$ of this event as

$$f_t = P\{Z_t \in A, S_1 > t\} + \sum_{n \in \mathbb{N}} \mathbb{E}[\mathbb{1}_{\{Z_t \in A\}} \mathbb{1}_{[S_n, S_{n+1})}(t)].$$

Using the tower property of conditional expectation and the regenerative property of Z, we can write

$$\mathbb{E}[\mathbb{1}_{\{Z_t \in A\}} \mathbb{1}_{[S_n, S_{n+1})}(t)] = \mathbb{E}[\mathbb{1}_{\{S_n \leqslant t\}} \mathbb{E}[\mathbb{1}_{\{Z_{S_n+t-S_n} \in A\}} \mathbb{1}_{[0, X_{n+1})}(t - S_n) \mid \mathcal{F}_{S_n}]] = \mathbb{E}[\mathbb{1}_{\{S_n \leqslant t\}} K^2(t - S_n)].$$

The result follows from aggregating the results for all $n \in \mathbb{N}$, the fact that $m^D = \sum_{n \in \mathbb{N}} F_{S_n}$, and exchange of summation and derivative using monotone convergence theorem. ☐

Example 1.12 (Age and excess time processes). Consider age and excess time processes associated with a delayed renewal process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, with distribution G for first inter renewal time and distribution F for subsequent inter renewal times. The nth segment corresponding to the age and excess time processes are given by

$$\zeta_n \triangleq (X_n, (t : t \in [0, X_n))), \qquad \eta_n \triangleq (X_n, (X_n - t : t \in [0, X_n))).$$

Since S is a delayed renewal sequence, segments are independent and identically distributed for $n \geqslant 2$. It follows that age and excess times are delayed regenerative processes. We fix a measurable set $B \triangleq [x, \infty)$ and compute the kernel functions K^1, K^2 for the delayed regenerative process A, as

$$K_t^1 \triangleq P\{A_t \geqslant x, S_1 > t\} = 1_{\{t \geqslant x\}} \bar{G}_t, \qquad K_t^2 \triangleq P\{A_{S_1+t} \geqslant x, S_1 + t \in [0, X_2)\} = 1_{\{t \geqslant x\}} \bar{F}_t.$$

Therefore, we can write the distribution of last renewal time for the delayed renewal process as

$$P\{S_{N_t} \leqslant x\} = P\{A_t \geqslant t - x\} = \mathbb{1}_{\{x \geqslant 0\}} \bar{G}_t + \int_0^{t \wedge x} dm_y^D \bar{F}_{t-y}.$$

1.3 Key Lemma

Theorem 1.13 (Key Lemma). *Consider a renewal sequence S with* i.i.d. *inter renewal times X having common distribution function F with positive mean, associated counting process N, and the renewal function m. Then, for all $0 \leqslant s \leqslant t$,*

$$P\{S_{N_t} \leqslant s\} = \bar{F}_t + \int_0^s dm_y \bar{F}_{t-y}.$$

Proof. We can see that the event $\{S_{N_t} \leqslant s\}$ that the time of last renewal prior to t is smaller than another time s can be partitioned into disjoint events corresponding to number of renewals until time t. Each of these disjoint events is equivalent to the occurrence of nth renewal before time s and the occurrence of $(n+1)$th renewal past time t. Since N_t is finite almost surely for $\mathbb{E}X_1 > 0$, we have the following almost sure equality

$$\{S_{N_t} \leqslant s\} = \bigcup_{n \in \mathbb{Z}_+} \{S_{N_t} \leqslant s, N_t = n\} = \bigcup_{n \in \mathbb{Z}_+} \{S_n \leqslant s, S_{n+1} > t\}.$$

We recognize that $S_0 = 0, S_1 = X_1$, and $S_{n+1} = S_n + X_{n+1}$. From linearity of expectation, monotone convergence theorem to exchange expectation and infinite summation of nonnegative random variables, and tower property of conditional expectation, we can write

$$P\{S_{N_t} \leqslant s\} = P\{X_1 > t\} + \sum_{n \in \mathbb{N}} \mathbb{E}[\mathbb{1}_{\{S_n \leqslant s\}} \mathbb{E}[\mathbb{1}_{\{X_{n+1} > t - S_n\}} | \sigma(S_n)]].$$

We recall that X is *i.i.d.* with distribution F and the distribution of nth renewal instant S_n is the n-fold convolution of F denoted by F_n. Taking expectation of $\bar{F}(t - S_n)\mathbb{1}_{\{S_n \leqslant s\}}$, we get

$$P\{S_{N_t} \leqslant s\} = \bar{F}(t) + \sum_{n \in \mathbb{N}} \int_{y=0}^{s} \bar{F}(t-y) dF_n(y).$$

Using monotone convergence theorem to interchange integral, summation, and derivatives, and noticing that $m(y) = \sum_{n \in \mathbb{N}} F_n(y)$, the result follows. ☐

Remark 5. Key lemma tells us that distribution of S_{N_t} has probability mass at 0 and density between $(0, t]$, that is,

$$P\{S_{N_t} = 0\} = \bar{F}(t), \qquad dF_{S_{N_t}}(y) = \bar{F}(t-y) dm(y), \quad 0 < y \leqslant t.$$

Probability of nth renewal taking place in the duration $[y, y+dy]$ is given by $P\{S_n \in (y, y+dy)\} = dF_n(y)$. Therefore, probability of some renewal taking place in the infinitesimal neighborhood of y, is

$$P\Big(\cup_{n \in \mathbb{N}} \{S_n \in (y, y+dy)\} \Big) = \sum_{n \in \mathbb{N}} dF_n(y) = dm(y).$$

Probability of no renewal in the interval $(y + dy, t]$, given the nth renewal occurred at time y, is given by $P\{X_{n+1} > t - y\} = \bar{F}(t-y)$. It follows that

$$P\{\text{renewal occurs in } (y, y+dy) \text{ and next arrival after } t - y\} = dF_{S_{N_t}}(y).$$

That is, the density of last renewal time S_{N_t} has the interpretation of renewal taking place in the infinitesimal neighborhood of y, and no renewal in the duration $[y, t]$.

Example 1.14 (Poisson process). Consider a renewal sequence S with *i.i.d.* inter renewal time sequence X distributed exponentially with rate $\lambda > 0$. Then, the distribution of last renewal is given by

$$P\{S_{N_t} \leqslant x\} = e^{-\lambda t} + \int_0^x \lambda e^{-\lambda(t-y)} dy = e^{-\lambda(t-x)}, \qquad 0 \leqslant x \leqslant t.$$

Exercise 1.15. Find the age and the excess time distribution for a Poisson process.

Corollary 1.16 (Delayed Key Lemma). *Consider a delayed renewal process S, with distribution G for first inter renewal time, distribution F for subsequent inter renewal times, associated counting process N^D, and the renewal function m^D. Then, for all $0 \leqslant s \leqslant t$,*

$$P\left\{S_{N_t^D} \leqslant s\right\} = \bar{G}_t + \int_0^s \bar{F}_{t-y} dm_y^D.$$