Lecture-11: Key Renewal Theorem

1 Blackwell Theorem

Lemma 1.1. *Consider a renewal sequence S with inter renewal time distribution F and renewal function m. If* $\inf\{x \in \mathbb{R}_+ : F(x) = 1\} = \infty$, *then* $\sup\{m_t - m_{t-b} : t \in \mathbb{R}_+\} < \infty$ *for any* $b > 0$.

Proof. Recall that $m = \sum_{n\in\mathbb{N}} F_n$ and hence $m * F = m - F$. This implies that $m * (1 - F) = F$. Since the function $1 - F$ is monotonically non increasing, $\inf_{s\in[0,b]} \bar{F}(s) = \bar{F}(b)$. Therefore,

$$1 \geqslant F(t) = \int_0^t dm_s \bar{F}(t-s) \geqslant \int_{t-b}^t dm_s \bar{F}(t-s) \geqslant (m_t - m_{t-b})\bar{F}(b).$$

Since $F(b) < 1$ for any $b \in \mathbb{R}_+$, we obtain the result. □

Theorem 1.2 (Blackwell's Theorem). *Consider a renewal sequence S with renewal function m, and* i.i.d. *inter renewal time sequence X with common distribution F such that* $\inf\{x \in \mathbb{R}_+ : F(x) = 1\} = \infty$.
(a) If the renewal sequence is aperiodic, then $\lim_{t\to\infty} m_{t+a} - m_t = a/\mathbb{E}X_1$ *for all* $a \geqslant 0$.
(b) If the renewal sequence has period d, then $\lim_{n\to\infty} m_{(n+1)d} - m_{nd} = d/\mathbb{E}X_1$.

Proof. We will show later that the following limit exists for aperiodic renewal sequences

$$g(a) \triangleq \lim_{t\to\infty} [m_{t+a} - m_t] \tag{1}$$

(a) However, we show that if this limit does exist, it is equal to $a/\mathbb{E}X_1$ as a consequence of elementary renewal theorem. To this end, note that $m_{t+a+b} - m_t = m_{t+a+b} - m_{t+a} + m_{t+a} - m_t$. Taking limits on both sides of the above equation, we conclude that $g(a+b) = g(a) + g(b)$. The only nondecreasing solution of such a $g \in \mathbb{R}_+^{\mathbb{R}_+}$ is $g(a) = ca$, for all $a \in \mathbb{R}_+$ and some positive constant c. To show $c = 1/\mathbb{E}X_1$, we define a sequence $x \in \mathbb{R}_+^{\mathbb{N}}$ in terms of renewal function m_t for each $n \in \mathbb{N}$, as

$$x_n \triangleq m_n - m_{n-1}.$$

Note that $\sum_{i=1}^n x_i = m_n$ and $\lim_{n\in\mathbb{N}} x_n = g(1) = c$. Further recall that, if a sequence $x \in \mathbb{R}^{\mathbb{N}}$ converges, then the running average sequence $a \in \mathbb{R}^{\mathbb{N}}$ defined as $a_n \triangleq \frac{1}{n}\sum_{i=1}^n x_i$ converges to the same limit. Hence, we have the Cesàro mean converging to $\lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=1}^n x_i = \lim_{n\in\mathbb{N}} \frac{m_n}{n} = c$. Therefore, we can conclude $c = 1/\mathbb{E}X_1$ by elementary renewal theorem.

(b) If the renewal sequence is periodic with period d, the limit in (1) doesn't exist, as shown in the following example. However, the theorem is true for lattice again by elementary renewal theorem. We can define $x_n \triangleq m_{nd} - m_{(n-1)d}$ and observe that $\sum_{i=1}^n x_i = m_{nd}$ and the Cesàro mean $\frac{1}{n}\sum_{i=1}^n x_i$ converges to $d/\mathbb{E}X_1$ by elementary renewal theorem.

□

Example 1.3. Consider a renewal sequence with $P\{X_1 = 1\} = 1$, that is, there is a renewal at every positive integer time instant with unit probability. We observe that it is a periodic renewal sequence with period $d = 1$. Now, for $a = 0.25$, and $t_n = n + (-1)^n a$, we see that $m_{t_n} = N_{t_n} = n - \mathbb{1}_{\{n \text{ odd}\}}$, and $m_{t_n+a} = n$. It follows that $m_{t_n+a} - m_{t_n} = \mathbb{1}_{\{n \text{ odd}\}}$, and hence $\lim_{t_n\to\infty} m_{t_n+a} - m_{t_n}$ does not exist. It follows that $\lim_{t\to\infty} m_{t+a} - m_t$ does not exist.

Exercise 1.4. Let m be the renewal function associated with an aperiodic renewal sequence. Show that the following limit exists
$$g(a) \triangleq \lim_{t\to\infty}[m_{t+a} - m_t].$$

Remark 1. For a renewal sequence with positive periodicity $d > 0$, there can be no more than one renewal at each time instant nd. In this case,
$$\lim_{n\to\infty} P\{\text{renewal at } nd\} = d/\mathbb{E}X_1.$$

Corollary 1.5 (Delayed Blackwell's Theorem). *Consider an aperiodic and recurrent delayed renewal process S with independent inter renewal times X with first inter renewal time distribution G and common inter renewal time distribution F for $(X_n : n \geqslant 2)$ such that* $\inf\{x \in \mathbb{R}_+ : F(x) = 1\} = \infty$. *We denote the associated renewal function as m^D.*

(a) If the renewal sequence is aperiodic, then $\lim_{t\to\infty} m^D_{t+a} - m^D_t = a/\mathbb{E}X_2$ for all $a \geqslant 0$.

(b) If renewal sequence has period d, then $\lim_{n\to\infty} m^D_{(n+1)d} - m^D_{nd} = d/\mathbb{E}X_2$.

2 Key Renewal Theorem

Theorem 2.1 (Key renewal theorem). *Consider a recurrent renewal process S with* i.i.d. *inter renewal time sequence X having common distribution F and finite mean, associated renewal function m, and a directly Riemann integrable function $z \in \mathbb{D}$.*

(a) If renewal sequence is aperiodic, then $\lim_{t\to\infty} \int_0^t z_{t-x} dm_x = \frac{1}{\mathbb{E}X_1}\int_0^\infty z_t dt$.

(b) If renewal sequence has period d, then $\lim_{n\to\infty} \int_0^{nd} z_{nd-x} dm_x = \frac{d}{\mathbb{E}X_1}\sum_{k\in\mathbb{Z}_+} z_{kd}$.

Proposition 2.2 (Equivalence). *Blackwell's theorem and key renewal theorem are equivalent.*

Proof. Let's assume key renewal theorem is true. We fix $a > 0$ and select a simple function $z \in \mathbb{R}_+^{\mathbb{R}_+}$ as an indicator for the interval $[0,a]$, i.e. $z_t \triangleq \mathbb{1}_{[0,a]}(t)$ for any $t \in \mathbb{R}_+$, and $z \in \mathbb{D}$ from Proposition A.4.

(a) Let S be an aperiodic renewal sequence, then from Key Renewal Theorem, we have $\lim_{t\to\infty}[m_t - m_{t-a}] = a/\mathbb{E}X_1$.

(b) Let the period of renewal sequence S be d, then from Key Renewal Theorem, we have for $\lim_{n\to\infty}[m_{nd-a} - m_{nd}] = d/\mathbb{E}X_1$ for $a < d$. In this case, we have $m_{nd-a} = m_{(n-1)d}$ and the result follows.

For the converse, we assume that Blackwell's theorem holds true.

(a) We defer the formal proof of converse for an aperiodic renewal sequence to a later stage. We observe that, from Blackwell theorem, it follows
$$\lim_{t\to\infty}\frac{dm(t)}{dt} \overset{(a)}{=} \lim_{a\to 0}\lim_{t\to\infty}\frac{1}{a}(m_{t+a} - m_t) = 1/\mathbb{E}X_1.$$
where in (a) we can exchange the order of limits under certain regularity conditions.

(b) When renewal sequence has period d, then dm_x is an impulse at multiple of d, and $\int_0^{nd} z_{nd-x} dm_x = \sum_{k=0}^n z_{(n-k)d}(m_{(k+1)d} - m_{kd})$. The result follows from exchange of limits for dRI $z \in \mathbb{D}$.

□

Remark 2. Key renewal theorem is very useful in computing the limiting value of some function g, where g_t is a probability or expectation of an event at an arbitrary time t, for a regenerative process. This value is computed by conditioning on the time of last regeneration prior to time t.

Corollary 2.3 (Delayed key renewal theorem). *Consider a recurrent delayed renewal process S with independent inter renewal times X with first inter renewal time distribution G and common inter renewal time distribution F for $(X_n : n \geqslant 2)$. Let the renewal function be denoted by m^D, $z \in \mathbb{D}$ be a directly Riemann integrable function.*

(a) If delayed renewal sequence is aperiodic, then $\lim_{t\to\infty} \int_0^t z_{t-x} dm_x = \frac{1}{\mathbb{E}X_2}\int_0^\infty z_t dt$.

(b) If delayed renewal sequence has period d, then $\lim_{n\to\infty} \int_0^{nd} z_{nd-x} dm_x = \frac{d}{\mathbb{E}X_2}\sum_{k\in\mathbb{Z}_+} z_{kd}$.

Remark 3. Any kernel function $t \mapsto K_t \triangleq P\{z_t \in A, X_1 > t\} \leqslant \bar{F}(t)$, and hence $K \in \mathbb{D}$ from Proposition A.4(b).

A Directly Riemann Integrable

For each scalar $h > 0$ and natural number $n \in \mathbb{N}$, we can define intervals $I_n(h) \triangleq [(n-1)h, nh)$, such that the collection $(I_n(h), n \in \mathbb{N})$ partitions the positive real-line $\mathbb{R}_+$. Consider a function $z \in \mathbb{R}_+ \to \mathbb{R}_+$ bounded over finite intervals, then we can denote the infimum and supremum of z in the interval I_n as

$$\underline{z}_n^h \triangleq \inf\{z_t : t \in I_n(h)\} \qquad \overline{z}_n^h \triangleq \sup\{z_t : t \in I_n(h)\}.$$

We can define functions $\underline{z}_h, \overline{z}_h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\underline{z}_t^h \triangleq \sum_{n\in\mathbb{N}} \underline{z}_n^h \mathbb{1}_{I_n(h)}(t)$ and $\overline{z}_t^h \triangleq \sum_{n\in\mathbb{N}} \overline{z}_n^h \mathbb{1}_{I_n(h)}(t)$ for all $t \in \mathbb{R}_+$. From the definition, we have $\underline{z}_h \leqslant z \leqslant \overline{z}_h$ for all $h \geqslant 0$. The infinite sums of infimum and supremums over all the intervals $(I_n(h), n \in \mathbb{N})$ are denoted by

$$\int_{t\in\mathbb{R}_+} \underline{z}_t^h dt = h \sum_{n\in\mathbb{N}} \underline{z}_n^h, \qquad \int_{t\in\mathbb{R}_+} \overline{z}_t^h dt = h \sum_{n\in\mathbb{N}} \overline{z}_n^h.$$

Remark 4. Since $\underline{z}_h \leqslant z \leqslant \overline{z}_h$, we observe that $\int_{t\in\mathbb{R}_+} \underline{z}_t^h dt \leqslant \int_{t\in\mathbb{R}_+} \overline{z}_t^h dt$. We observe that $\underline{z}_h$ and $\overline{z}_h$ are nondecreasing and nonincreasing in h respectively. As as $h \downarrow 0$, if both left and right limits exist and are equal, then the integral value $\int_{t\in\mathbb{R}_+} z_t dt$ is equal to the limit.

Definition A.1 (directly Riemann integrable (d.R.i.)). A function $z : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is *directly Riemann integrable* and denoted by $z \in \mathbb{D}$ if the partial sums obtained by summing the infimum and supremum of h, taken over intervals obtained by partitioning the positive axis, are finite and both converge to the same limit, for all finite positive interval lengths. That is, $\sum_{n\in\mathbb{N}} h\overline{z}_n^h < \infty$ and $\lim_{h\downarrow 0} \int_{t\in\mathbb{R}_+} \overline{z}_t^h dt = \lim_{h\downarrow 0} \int_{t\in\mathbb{R}_+} \underline{z}_t^h dt$. The limit is denoted by

$$\int_{t\in\mathbb{R}_+} z_t dt = \lim_{h\downarrow 0} \sum_{n\in\mathbb{N}} h\overline{z}_n^h = \lim_{h\downarrow 0} \sum_{n\in\mathbb{N}} h\underline{z}_n^h.$$

For a real function $z \in \mathbb{R}^{\mathbb{R}_+}$, we can define the positive and negative parts by $z^+, z^- \in \mathbb{R}_+^{\mathbb{R}_+}$ such that $z_t^+ \triangleq z_t \vee 0$, and $z_t^- \triangleq -(z_t \wedge 0)$ for all $t \in \mathbb{R}_+$. If both $z^+, z^- \in \mathbb{D}$, then $z \in \mathbb{D}$ and the limit is

$$\int_{\mathbb{R}_+} z_t dt \triangleq \int_{\mathbb{R}_+} z_t^+ dt - \int_{\mathbb{R}_+} z_t^- dt.$$

Remark 5. We compare the definitions of directly Riemann integrable and Riemann integrable functions. For a finite positive M, a function $z \in \mathbb{R}^{[0,M]}$ is Riemann integrable if $\lim_{h\downarrow 0} \int_0^M \overline{z}_t^h dt = \lim_{h\downarrow 0} h \int_0^M \underline{z}_t^h dt$. In this case, the limit is the value of the integral $\int_0^M z_t dt$. For a function $z \in \mathbb{R}^{\mathbb{R}_+}$, $\int_{t\in\mathbb{R}_+} z_t dt = \lim_{M\to\infty} \int_0^M z_t dt$, if the limit exists. For many functions, this limit may not exist.

Remark 6. A directly Riemann integrable function over $\mathbb{R}_+$ is also Riemann integrable, but the converse need not be true.

Lemma A.2. *We define a map $z \in \mathbb{R}_+^{\mathbb{R}_+}$ as $z_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{E_n}(t)$ for all $t \in \mathbb{R}_+$, where $E_n \triangleq \left[n - \frac{1}{2n^2}, n + \frac{1}{2n^2}\right]$ for each $n \in \mathbb{N}$. Then, z is Riemann integrable, but not directly Riemann integrable.*

Proof. We will show that z is Riemann integrable, however $\int_{t\in\mathbb{R}_+} \overline{z}_t dt$ is always infinite.

(a) We observe that $\int_{\mathbb{R}_+} \mathbb{1}_{E_n}(t) dt = 1/n^2$ and hence $\sum_{n\in\mathbb{N}} \int_{\mathbb{R}_+} \mathbb{1}_{E_n}(t) dt = \sum_{n\in\mathbb{N}} 1/n^2 < \infty$. Interchanging infinite sum and integral for positive terms by monotone convergence theorem, we obtain $\int_{t\in\mathbb{R}_+} z_t dt < \infty$, and hence z is Riemann integrable.

(b) It suffices to show that $\overline{z}_m(h) = 1$ for all $m \in \mathbb{N}$. Since the collection $(I_n(h) : n \in N)$ partitions the entire $\mathbb{R}_+$, for each $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that $E_n \cap I_m(h) \neq \emptyset$, and the result follows. □

Exercise A.3 (Necessary conditions for d.R.i.). If a function $z : \mathbb{R}_+ \to \mathbb{R}_+$ is directly Riemann integrable, then show that z is bounded and continuous a.e.

Exercise A.4 (Sufficient conditions for d.R.i.). Show that if any of the following conditions hold for a function $z : \mathbb{R}_+ \to \mathbb{R}_+$, then it is directly Riemann integrable.

(a) z is monotone nonincreasing, and Lebesgue integrable.

(b) z is bounded above by a directly Riemann integrable function.

(c) z has bounded support.

(d) z is continuous, and has finite support.

(e) z is continuous, bounded, and $\int_{t\in\mathbb{R}_+} \bar{z}_t^h dt$ is bounded for some $h > 0$.

(f) $\int_{t\in\mathbb{R}_+} \bar{z}_t^h dt$ is bounded for some $h > 0$.

Exercise A.5. For any directly Riemann integrable function $z : \mathbb{R}_+ \to \mathbb{R}_+$ show that $\lim_{t\to\infty} z_t = \lim_{n\to\infty} \bar{z}_n^h$.

Proposition A.6 (Tail Property). *If $z : \mathbb{R}_+ \to \mathbb{R}_+$ is directly Riemann integrable and has bounded integral value, then $\lim_{t\to\infty} z_t = 0$.*

Proof. If $z \in \mathbb{D}$, then $h\sum_{n\in\mathbb{N}} \bar{z}_n^h < \infty$ for all $h > 0$. This implies that the infinite positive sum $\sum_n \bar{z}_n^h$ is finite, and hence $\lim_{n\to\infty} \bar{z}_n^h = \lim_{t\to\infty} z_t = 0$. ☐

Corollary A.7. *Any distribution $F : \mathbb{R}_+ \to [0,1]$ with finite mean μ, the complementary distribution function $\bar{F}$ is d.R.i.*

Proof. Since $\bar{F}$ is monotonically nonincreasing and its Lebesgue integration is $\int_{\mathbb{R}_+} \bar{F}(t)dt = \mu$, the result follows from Proposition A.4(a). ☐