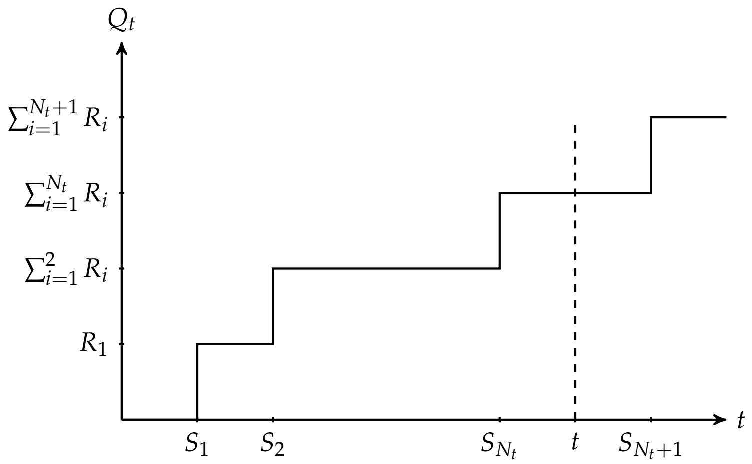
Lecture-12: Renewal reward process

1 Renewal reward process

Definition 1.1 (Renewal reward process). Consider a renewal sequence S with *i.i.d.* inter renewal times X having common distribution F, and the associated counting process N. At the end of each renewal interval $n \in \mathbb{N}$, a random reward R_n is earned at time S_n, where the reward R_n is possibly dependent on the duration X_n, but is *i.i.d.* across intervals $n \in \mathbb{N}$. We assume $(X, R) : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ to be an *i.i.d.* sequence, then R is called *reward sequence* associated with renewal sequence S, and the *reward process* $Q : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ is defined as the accumulated reward earned until each time $t \in \mathbb{R}_+$, denoted as $Q_t \triangleq \sum_{i=1}^{N_t} R_i$. The reward rate at time $t \in \mathbb{R}_+$ is defined as $\frac{Q_t}{t}$.

Definition 1.2 (Next reward process). For a renewal reward sequence (X, R), we define the reward to earn at next renewal instant as R_{N_t+1}, its mean as $g_t \triangleq \mathbb{E}R_{N_t+1}$, and kernel as $K_t \triangleq \mathbb{E}R_{N_t+1}\mathbb{1}_{\{X_1>t\}}$ at each time $t \in \mathbb{R}_+$.



Example 1.3. Consider a renewal sequence S with *i.i.d.* inter-renewal time sequence X, and associated counting process N. Consider an *i.i.d.* renewal sequence R defined as $R_n \triangleq 1$ for each $n \in \mathbb{N}$. Then the reward process Q is the same as the counting process N.

Example 1.4. Consider a renewal sequence S with *i.i.d.* inter-renewal time sequence X. Consider an *i.i.d.* renewal sequence R defined as $R_n \triangleq X_n$ for each $n \in \mathbb{N}$. Then the reward process Q is the last renewal instant $Q_t = S_{N_t}$ at any time $t \in \mathbb{R}_+$.

Lemma 1.5 (Next reward). *Consider a renewal reward sequence (X, R) with renewal sequence S, inter renewal time distribution F, renewal function m, and the associated next reward process. If $\mathbb{E}X_1 < \infty$ and $\mathbb{E}|R_1| < \infty$, then the mean g and kernel K of next reward process are related as $g = (1 + m) * K$.*

Proof. The nth segment for next reward process is $\zeta_n = (X_n, R_n)$. It follows that the segment sequence $\zeta : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ is *i.i.d.* , and hence R_{N_t+1} is regenerative process with regeneration intervals being the renewal intervals $[S_{n-1}, S_n)$. Considering the event of no renewal or at least one renewal before time t for the regenerative process R_{N_t+1}, we can write at each $t \in \mathbb{R}_+$

$$g_t = \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1>t\}}] + \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1\leqslant t\}}], \qquad K_t = \mathbb{E}[R_1\mathbb{1}_{\{S_1>t\}}].$$

From the regenerative property of R_{N_t+1}, we obtain $\mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1\leqslant t\}} \mid \mathcal{F}_{S_1}] = \mathbb{1}_{\{S_1\leqslant t\}}g_{t-S_1}$. Together with the tower property of conditional expectation, we have at each $t \in \mathbb{R}_+$

$$\mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1\leqslant t\}}] = \mathbb{E}[\mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1\leqslant t\}} \mid \mathcal{F}_{S_1}] = \mathbb{E}[\mathbb{1}_{\{S_1\leqslant t\}}g_{t-S_1}].$$

Combining the two case, we get the renewal equation $g = K + g * F$. Applying renewal theorem to this renewal equation for g with kernel K and inter renewal time distribution F, we obtain the result. ☐

Corollary 1.6. *If* $\mathbb{E}X_1 < \infty$ *and* $\mathbb{E}|R_1| < \infty$, *then* $\lim_{t\to\infty}\frac{1}{t}g_t = 0$.

Proof. From Lemma 1.5, we have $g = (1+m) * K$ where m is the renewal function and K is the kernel function. Applying the conditional Jensen's inequality to convex function absolute, we observe that the kernel function K is bounded above as

$$K_t \triangleq \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1>t\}}] = \mathbb{E}[\mathbb{E}[R_1\mathbb{1}_{\{X_1>t\}} \mid \sigma(X_1)]] \leqslant \mathbb{E}[\mathbb{E}[|R_1|\,\mathbb{1}_{\{X_1>t\}} \mid \sigma(X_1)]].$$

From finiteness of $\mathbb{E}|R_1|$, it follows that $\lim_{t\to\infty}K_t = 0$. We fix $\epsilon > 0$ and select T such that $|K_u| \leqslant \epsilon$ for all $u \geqslant T$. Since $g = (1+m) * K$, we apply triangle inequality to obtain for all $t \geqslant T$,

$$\frac{|g_t|}{t} \leqslant \frac{|K_t|}{t} + \int_0^{t-T}\frac{|K_{t-u}|}{t}dm_u + \int_{t-T}^{t}\frac{|K_{t-u}|}{t}dm_u \leqslant \frac{\epsilon}{t} + \frac{\epsilon m_{t-T}}{t} + \mathbb{E}|R_1|\frac{(m_t - m_{t-T})}{t}.$$

Taking limits and applying elementary renewal and Blackwell's theorem, we get $\limsup_{t\to\infty}\frac{|g_t|}{t} \leqslant \frac{\epsilon}{\mathbb{E}X_1}$. The result follows since $\epsilon > 0$ was arbitrary. ☐

Theorem 1.7 (Renewal reward). *Consider a renewal reward sequence* (X, R) *and the associated reward process* Q. *If* $0 < \mathbb{E}X_1 < \infty$ *and* $\mathbb{E}|R_1| < \infty$, *then the reward rate converges almost surely and in mean, i.e.*

$$\lim_{t\to\infty}\frac{1}{t}Q_t = \frac{\mathbb{E}R_1}{\mathbb{E}X_1}\ a.s.\ , \qquad \lim_{t\to\infty}\frac{1}{t}\mathbb{E}Q_t = \frac{\mathbb{E}R_1}{\mathbb{E}X_1}.$$

Proof. We will first show the almost sure convergence and then the convergence in mean.

(a) From the strong law of large numbers and strong law for counting processes, we have the following almost sure equalities

$$\lim_{t\to\infty}\frac{Q_t}{N_t} = \lim_{t\to\infty}\frac{1}{N_t}\sum_{i=1}^{N_t}R_i = \mathbb{E}R_1, \qquad \lim_{t\to\infty}\frac{1}{t}N_t = \frac{1}{\mathbb{E}X_1}.$$

The almost sure convergence follows from writing the reward rate as $\frac{1}{t}Q_t = (Q_t/N_t)(N_t/t)$.

(b) Let $\mathcal{F}_\bullet$ be the natural filtration associated with the renewal reward sequence such that for each $n \in \mathbb{N}$, we have $\mathcal{F}_n \triangleq \sigma(X_1, R_1, \dots, X_n, R_n)$. Then $N_t + 1$ is a stopping time adapted to $\mathcal{F}_\bullet$ where almost sure finiteness follows from the fact that $\mathbb{E}X_1 > 0$ and adaptation is clear. Hence, it follows from Wald's lemma and the definition of mean of next reward process, that

$$\mathbb{E}Q_t = \mathbb{E}\sum_{i=1}^{N_t}R_i = \mathbb{E}\sum_{i=1}^{N_t+1}R_i - \mathbb{E}R_{N_t+1} = (m_t+1)\mathbb{E}R_1 - g_t.$$

The result follows by dividing both sides of the equation by $t > 0$, taking limit $t \to \infty$, and applying elementary renewal theorem and Corollary 1.6.

☐

Corollary 1.8. *Renewal reward theorem applies to a reward process* Q *that accrues positive reward continuously over a renewal duration. The total reward in a renewal duration* X_n *remains* R_n *as before, with the sequence*(X, R) *being* i.i.d. .

Proof. Let $Q : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ be the accumulated reward process when the reward accrual is positive and continuous in time, with Q_t denoting the accumulated reward until time $t \in \mathbb{R}_+$. We define the total accumulated reward in the nth renewal interval as $R_n \triangleq Q_{S_n} - Q_{S_{n-1}} > 0$. From positivity of reward accrual rate, it follows that

$$\frac{\sum_{n=1}^{N_t}R_n}{t} \leqslant \frac{Q_t}{t} \leqslant \frac{\sum_{n=1}^{N_t+1}R_n}{t}.$$

Almost sure convergence of reward rate follows from application of strong law of large numbers. For the convergence in mean, we observe that at each $t \in \mathbb{R}_+$

$$(m_t+1)\mathbb{E}R_1 - \mathbb{E}R_{N_t+1} \leqslant \mathbb{E}Q_t \leqslant (m_t+1)\mathbb{E}R_1.$$

The second result follows from Corollary 1.6 that shows $\lim_{t\to\infty}\frac{1}{t}\mathbb{E}R_{N_t+1} = 0$. ☐

1.1 Inspection Paradox

Definition 1.9. Consider a renewal sequence S with *i.i.d.* inter renewal time sequence X and associated counting process N. At each time $t \in \mathbb{R}_+$, we define the current renewal interval length as $X_{N_t+1} \triangleq Y_t + A_t$, its complementary distribution as $g_t^x \triangleq P\{X_{N_t+1} > x\}$, and kernel map $k_t^x \triangleq P\{X_{N_t+1} > x, S_1 > t\}$ for each $x \in \mathbb{R}_+$.

Lemma 1.10 (Inspection Paradox). *Consider a renewal sequence S with* i.i.d. *inter renewal time sequence X and associated counting process N. If $0 < \mathbb{E}X_1 < \infty$, then $\mathbb{E}X_{N_t+1} \geqslant \mathbb{E}X_1$.*

Proof. From the definition of expectation of positive random variables, it suffices to show that $g_t^x \geqslant \bar{F}(x)$ for each $x, t \in \mathbb{R}_+$. We will show this in steps.

Step 1. We first observe that X_{N_t+1} is a regenerative process with regeneration instant sequence S since its segment during the nth renewal period $[S_{n-1}, S_n)$ is $\xi_n \triangleq (X_n, (X_n, t \in [S_{n-1}, S_n)))$.

Step 2. From the definition of kernel function, we have $k_t^x = \bar{F}(x \vee t)$ for each $t, x \in \mathbb{R}_+$. From the regenerative property of the length of the current renewal period, we obtain the renewal equation $g_t^x = k_t^x + \mathbb{E}g_{t-S_1}^x \mathbb{1}_{\{S_1 \leqslant t\}}$, i.e. $g^x = k^x + g^x * F$. Applying renewal theorem for finite mean inter renewal times X, we can write the unique solution to the renewal equation as $g^x = (1+m) * k^x$.

Step 3. We define nondecreasing functions $z \mapsto f(z) \triangleq \mathbb{1}_{\{z>x\}}$ and $z \mapsto g(z) \triangleq \mathbb{1}_{\{z>t\}}$. Applying the Chebyshev's inequality to f, g, and random variable X_1, we obtain

$$k_t^x = \mathbb{E}\mathbb{1}_{\{X_{N_t+1} > x, X_1 > t\}} = \mathbb{E}\mathbb{1}_{\{X_1 > x, X_1 > t\}} \geqslant \bar{F}(x)\bar{F}(t).$$

Taking convolution with $1+m$ on both sides of the above equation, and observing that $g^x = (1+m) * k^x$ and $(1+m) * \bar{F} = 1$, we get the result.

□

Remark 1. The accumulated reward R_{N_t+1} in the current renewal interval is possibly dependent on the current renewal duration X_{N_t+1}. If the reward accrual rate is positive, then it follows from the inspection paradox that $\mathbb{E}R_{N_t+1} \geqslant \mathbb{E}R_1$.

Definition 1.11. Consider an *i.i.d.* renewal reward sequence (X, R) with renewal sequence S and counting process N. At each time $t \in \mathbb{R}_+$, we define the marginal tail probability of next reward as $f_t^x \triangleq P\{R_{N_t+1} > x\}$ and associated kernel map $k_t^x \triangleq P\{R_{N_t+1} > x, S_1 > t\}$ for each $x \in \mathbb{R}_+$.

Lemma 1.12. *For a renewal reward process with positive reward accrual rate, we have $\mathbb{E}R_{N_t+1} \geqslant \mathbb{E}R_1$.*

Proof. From the definition of expectation of positive random variables, it suffices to show that $f_t^x \geqslant P\{R_1 > x\}$ for each $x, t \in \mathbb{R}_+$. We will show this in steps.

Step 1. We recall that R_{N_t+1} is a regenerative process with regeneration instant sequence S. From the definition of kernel function, we have $k_t^x = P\{R_1 > x, X_1 > t\}$ for each $t, x \in \mathbb{R}_+$. From the regenerative property of the length of the current renewal period, we obtain the renewal equation $g_t^x = k_t^x + \mathbb{E}g_{t-S_1}^x \mathbb{1}_{\{S_1 \leqslant t\}}$, i.e. $g^x = k^x + g^x * F$. Applying renewal theorem for finite mean inter renewal times X, we can write the unique solution to the renewal equation as $g^x = (1+m) * k^x$.

Step 2. We define nondecreasing functions $z \mapsto f(z) \triangleq \mathbb{1}_{\{z>x\}}$ and $z \mapsto g(z) \triangleq \mathbb{1}_{\{z>t\}}$. Applying the Chebyshev's inequality to f, g, and random variable X_1, we obtain

$$k_t^x = \mathbb{E}\mathbb{1}_{\{R_{N_t+1} > x, X_1 > t\}} = \mathbb{E}\mathbb{1}_{\{R_1 > x, X_1 > t\}} \geqslant P\{R_1 > x\}\bar{F}(t).$$

Taking convolution with $1+m$ on both sides of the above equation, and observing that $g^x = (1+m) * k^x$ and $(1+m) * \bar{F} = 1$, we get the result.

□

A Chebyshev's sum inequality

Theorem A.1. *Consider two nondecreasing positive measurable functions $f, g : \mathbb{R} \to \mathbb{R}_+$ and a random variable $X : \Omega \to \mathbb{R}$. Then, $\mathbb{E}f(X)g(X) \geqslant \mathbb{E}f(X)\mathbb{E}g(X)$.*

Proof. Consider a random vector $Y : \Omega \to \mathbb{R}^2$ to be an *i.i.d.* replica of $X : \Omega \to \mathbb{R}$ and the product $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))$. From the linearity of expectation and Y_1, Y_2 being *i.i.d.* to X, we can expand the mean of the product as

$$\mathbb{E}(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2)) = 2\mathbb{E}f(X)g(X) - 2\mathbb{E}f(X)\mathbb{E}g(X) = 2\mathbb{E}(f(X) - \mathbb{E}f(X))(g(X) - \mathbb{E}g(X)).$$

Since f, g are nondecreasing, we have $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))\mathbb{1}_{\{Y_1 \geqslant Y_2\}} \geqslant 0$ and $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))\mathbb{1}_{\{Y_1 < Y_2\}} \geqslant 0$. Summing the two terms, we obtain

$$(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2)) \geqslant 0.$$

□