Lecture-13: Limiting marginal distribution

1 Applications of key renewal theorem

Theorem 1.1 (Limiting distribution of regenerative process). *Consider a delayed regenerative process $Z : \Omega \to \mathcal{Z}^{\mathbb{R}_+}$ over a delayed renewal sequence S with independent inter renewal time sequence X with first inter renewal time distribution G and subsequent inter renewal time distribution F. For a Borel measurable set $A \in \mathcal{B}(\mathcal{Z})$ and we define kernel function $K^2 \in [0,1]^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$ as $K^2_t \triangleq P\{Z_{S_1+t} \in A, X_2 > t\}$. If $\mathbb{E}X_2 < \infty$ and Z is aperiodic, then*

$$\lim_{t\to\infty} P\{Z_t \in A\} = \frac{1}{\mathbb{E}X_2}\int_{t=0}^{\infty} K^2_t dt.$$

Proof. From the definition of kernel function K^2, we obtain that $K^2_t \leqslant \bar{F}_t$ for any $A \in \mathcal{B}(\mathcal{Z})$ and $t \in \mathbb{R}_+$. Since $\bar{F}$ is monotone nonincreasing and Lebesgue integrable with integral $\int_{t\in\mathbb{R}_+} \bar{F}_t dt = \mathbb{E}X_2 < \infty$, it is directly Riemann integrable. Since $K^2 \leqslant \bar{F}$ is bounded above by a direct Riemann integrable function, it follows that K^2 is also directly Riemann integrable.

For delayed renewal sequence S, we denote the associated delayed renewal function by m^D, and for delayed regenerative process Z, we define marginal probability distribution f as $f_t \triangleq P\{Z_t \in A\}$ and kernel $K^1 \in [0,1]^{\mathbb{R}_+}$ as $K^1_t \triangleq P\{Z_t \in A, X_1 > t\}$ for each time $t \in \mathbb{R}_+$. We recall that $f = K^1 + K^2 * m^D$ and observe that $0 \leqslant K^1 \leqslant \bar{G}$. Applying Key renewal theorem to regenerative process Z and observing that $\lim_{t\to\infty} K^1_t = 0$, we get the limiting probability of the event $\{Z_t \in A\}$ as

$$\lim_{t\to\infty} f_t = \lim_{t\to\infty} (m^D * K^2)_t = \frac{1}{\mathbb{E}X_2}\int_{t=0}^{\infty} K^2_t dt.$$

□

1.1 Age-dependent branching process

Definition 1.2 (Age-dependent branching process). Consider a population, where each organism $i \in \mathbb{N}$ lives for an *i.i.d.* random time period of $T_i : \Omega \to \mathbb{R}_+$ units with common distribution function $F : \mathbb{R}_+ \to [0,1]$. Just before dying, each organism i produces an *i.i.d.* random number of offsprings $J_i : \Omega \to \mathbb{Z}_+$, with common distribution $p \in \mathcal{M}(\mathbb{N})$ and mean $n \triangleq \mathbb{E}[J_1] = \sum_{j\in\mathbb{Z}_+} jp_j$. We fix time $t \in \mathbb{R}_+$, and we denote the number of organisms alive at this time t as X_t and its mean as $m_t \triangleq \mathbb{E}X_t$. The stochastic process $X : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is called an *age dependent branching process*. We denote the history of this process until time t by $\mathcal{F}_t$.

Remark 1. This is a popular model in biology for population growth of various organisms. We are interested in computing the evolution of mean population m.

Lemma 1.3. *Consider the age dependent branching process X from Definition 1.2 starting from a single organism. The organism population regenerates into a random number of replicas at the end of life of each organism.*

Proof. Lifetime of each organism is an *i.i.d.* random variable and the number of their offsprings are *i.i.d.* . Hence, at the end of life of each organism, a probabilistic replica of the population is created. □

Definition 1.4. For age dependent branching process with *i.i.d.* lifetime distribution for each organism denoted by $F \in [0,1]^{\mathbb{R}_+}$ and mean number of offsprings n, we define α to be the unique solution to the equation

$$1 = n\int_0^{\infty} e^{-\alpha t} dF(t). \tag{1}$$

In terms of α and F, we define a distribution $G \in [0,1]^{\mathbb{R}_+}$ such that $dG(t) \triangleq ne^{-\alpha t}dF(t)$ for each $t \in \mathbb{R}_+$. We define the inter renewal time distribution $G_1 \triangleq G$ to inductively define the nth renewal time distribution $G_n \triangleq G_{n-1} * G$ and the associated renewal function $m^G \triangleq \sum_{n\in\mathbb{N}} G_n$.

Definition 1.5. For age dependent branching process X with mean m and common organism lifetime distribution F from Definition 1.2 and positive scalar α from Definition 1.4, we define a map $f \in [0,1]^{\mathbb{R}_+}$ and kernel $K \in [0,1]^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$ as

$$f_t \triangleq e^{-\alpha t} m_t, \qquad K_t \triangleq e^{-\alpha t} \bar{F}(t).$$

Theorem 1.6. *Consider the age dependent branching process X from Definition 1.2 with initial population $X_0 \triangleq 1$, distribution G and associated renewal function m^G from Definition 1.4, and maps f, K from Definition 1.5. If $\int_{t\in\mathbb{R}_+} tdG(t) < \infty$, then $f = (1 + m^G) * K$.*

Proof. Recall that $X_0 = 1$. Let T_1 and J_1 denote the life time and number of offsprings of the first organism. If $T_1 > t$, then X_t is still equal to $X_0 = 1$. In this case, we have

$$\mathbb{E}[X_t \mathbb{1}_{\{T_1 > t\}} \mid \mathcal{F}_{T_1}] = X_0 \mathbb{1}_{\{T_1 > t\}}. \tag{2}$$

If $T_1 \leqslant t$, then $X_{T_1} = J_1$ and each of the offsprings start the population growth, independent of the past, and stochastically identical to the population growth of the original organism starting at time T_1. That is, $(X^i_{T_1+u}, u \geqslant 0)$ is a stochastic replica of $(X_u, u \geqslant 0)$, and independent for each $i \in [J_1]$. Hence,

$$\mathbb{E}[X_t \mathbb{1}_{\{T_1 \leqslant t\}} \mid \mathcal{F}_{T_1}] = \mathbb{E}[\sum_{i=1}^{J_1} X^i_{t-T_1} \mathbb{1}_{\{T_1 \leqslant t\}} \mid \sigma(T_1)] = n m_{t-T_1} \mathbb{1}_{\{T_1 \leqslant t\}}. \tag{3}$$

Combining the case of number of organisms alive before first birth $\{T_1 > t\}$ from (2), and the case of number of organisms alive after first birth $\{T_1 \leqslant t\}$ from (3), we can write the mean function m_t as

$$m_t = \mathbb{E}[X_t \mathbb{1}_{\{T_1 > t\}}] + \mathbb{E}[X_t \mathbb{1}_{\{T_1 \leqslant t\}}] = \bar{F}(t) + n \int_0^t m_{t-u} dF(u). \tag{4}$$

This looks almost like a renewal function. We take positive scalar α and distribution G from Definition 1.4. Multiplying both sides of (4) by $e^{-\alpha t}$, substituting the definition of G, and substituting the definition of f, K from Definition 1.5, we get for each $t \in \mathbb{R}_+$

$$f_t = m_t e^{-\alpha t} = e^{-\alpha t} \bar{F}(t) + n \int_0^t e^{-\alpha(t-u)} m_{t-u} e^{-\alpha u} dF(u) = K_t + \int_0^t f_{t-u} dG_u.$$

That is, we can rewrite (4) as a renewal equation $f = K + f * G$ for map f in terms of kernel K and inter renewal distribution G. The result is the unique solution of this renewal equation. □

Corollary 1.7. *Consider the age dependent branching process X from Definition 1.2 with initial population $X_0 = 1$ and mean number of offspring n and organism lifetime distribution F. Let positive scalar α and G be from Definition 1.4. If $\int_{t\in\mathbb{R}_+} tdG(t) < \infty$, then $\lim_{t\to\infty} m_t e^{-\alpha t} = \frac{\frac{1}{\alpha}(1-\frac{1}{n})}{\int_0^\infty tdG(t)}$.*

Proof. From Theorem 1.6, we know that $f = (1 + m^G) * K$ where map $f_t = m_t e^{-\alpha t}$ and kernel function $K_t = e^{-\alpha t}\bar{F}(t)$ are defined for each $t \in \mathbb{R}_+$ in Definition 1.5, and m^G is the renewal function for inter renewal distribution G. Since $\bar{F}$ is monotone non increasing and Lebesgue integrable, it is directly Riemann integrable. From the definition of kernel function K and positivity of scalar α, we have $0 \leqslant K \leqslant \bar{F}$, and hence is directly Riemann integrable and $\lim_{t\to\infty} K_t = 0$. Hence, we can apply key renewal theorem to f to obtain the following limit

$$\lim_{t\to\infty} f_t = \lim_{t\to\infty} m_t e^{-\alpha t} = \frac{\int_{t\in\mathbb{R}_+} K_t dt}{\int_{t\in\mathbb{R}_+} tdG(t)}.$$

From the definition of kernel K, integration by parts, and definition of scalar α in (1), we can write the numerator as $\int_0^\infty e^{-\alpha t}\bar{F}(t)dt = \frac{1}{\alpha} - \frac{1}{\alpha}\int_0^\infty e^{-\alpha t} dF(t) = \frac{1}{\alpha}\left(1 - \frac{1}{n}\right)$. □

Remark 2. We conclude that the mean population is growing exponentially asymptotically, and $m_t \approx Ce^{\alpha t}$ for large time t, some constant C independent of time t, and a positive exponent α.

1.2 Alternating renewal processes

Definition 1.8 (Alternating renewal sequence). An *i.i.d.* random sequence $(Z, Y) : \Omega \to (\mathbb{R}_+^2)^{\mathbb{N}}$ is called an *alternating renewal sequence*, where Z_n and Y_n are nth *on* and *off* durations respectively. The on time duration Z_n and off time duration Y_n are not necessarily independent. We define *i.i.d.* inter renewal sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and corresponding renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ as $X_n \triangleq Z_n + Y_n$ and $S_n \triangleq \sum_{k=1}^{n} X_k$ for each $n \in \mathbb{N}$. We denote the associated counting process by N and the age process by A. We call the time interval $(S_{n-1}, S_{n-1} + Z_n]$ as nth on time followed by nth off time $(S_{n-1} + Z_n, S_n]$. We denote the distributions for on, off, and renewal periods by H, G, and F, respectively. When on and off times are independent, $F = H * G$.

Definition 1.9 (Alternating renewal process). For an alternating renewal sequence (Z, Y), we define an alternating stochastic process $W : \Omega \to \{0, 1\}^{\mathbb{R}_+}$ such that W_t indicates the renewal process S being in on state at time $t \in \mathbb{R}_+$. In particular, we can write alternating renewal process for any time $t \in \mathbb{R}_+$, as

$$W_t \triangleq \mathbb{1}_{\{A_t \leqslant Z_{N_t+1}\}}.$$

Remark 3. Alternating renewal processes form an important class of renewal processes, and model many interesting applications.

Lemma 1.10. *Alternating renewal process is a regenerative process.*

Proof. For each $n \in \mathbb{N}$, we observe that $W_{S_{n-1}+t} = \mathbb{1}_{(0,Z_n]}(t)$ for all $t \in (0, X_n]$. Hence, we define the nth segment

$$\zeta_n \triangleq (X_n, (\mathbb{1}_{(0,Z_n]}(t) : t \in (0, X_n]),$$

and observe that the segment sequence $(\zeta_n, n \in \mathbb{N})$ is *i.i.d.* . Therefore, it follows that the alternating renewal process W is regenerative over renewal sequence S. □

Theorem 1.11. *Consider the alternating renewal process W in Definition 1.9 over renewal sequence S and associated renewal function m. Let H be the distribution of* on *times, and we define* on *probability at any time $t \in \mathbb{R}_+$ as $P_t \triangleq P\{W(t) = 1\}$. If* i.i.d. *inter renewal times have finite mean and W is aperiodic, then $P = (1 + m) * \bar{H}$.*

Proof. In terms of the complementary distribution function $\bar{H}$ of the on times, we can compute the associated kernel function $K \in [0, 1]^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$, as

$$K_t \triangleq P\{W_t = 1, S_1 > t\} = P\{H_1 > t\} = \bar{H}_t.$$

Recall that alternating renewal process W is regenerative from Lemma 1.10. The result follows from the application of renewal theorem to regenerative process W with renewal function m for inter renewal times and kernel function K. □

Corollary 1.12. *Consider the alternating renewal sequence (Z, Y) in Definition 1.8 and alternating renewal process W in Definition 1.9. If* i.i.d. *inter renewal times have finite mean and W is aperiodic, then the limiting* on *probability is*

$$\lim_{t \to \infty} P_t = \frac{\mathbb{E}Z_1}{\mathbb{E}Z_1 + \mathbb{E}Y_1}.$$

Proof. Recall that the distribution of on times Z and off times Y are denoted by H and G respectively. From Theorem 1.11, the on probability at time $t \in \mathbb{R}_+$ is $P_t = \bar{H}_t + (m * \bar{H})_t$. Since $\mathbb{E}Z_1 < \infty$, we have $\lim_{t\to\infty} \bar{H}_t = 0$. Further, $\bar{H}$ is a directly Riemann integrable function since it is monotone nonincreasing and Lebesgue integrable to $\int_{t \in \mathbb{R}_+} \bar{H}_t dt = \mathbb{E}Z_1 < \infty$. Since $\mathbb{E}Z_1$ and $\mathbb{E}Y_1$ are finite and W is aperiodic, applying key renewal theorem to the limiting probability of alternating process being on, we get

$$\lim_{t \to \infty} P_t = \lim_{t \to \infty} (m * \bar{H})_t = \frac{1}{\mathbb{E}X_1} \int_{t \in \mathbb{R}_+} \bar{H}_t dt.$$

The result follows since $X_1 = Z_1 + Y_1$ and $\int_{t \in \mathbb{R}_+} \bar{H}_t dt = \mathbb{E}Z_1$. □

Exercise 1.13 (Age and excess time process as an alternating renewal process). Consider a renewal sequence with a non-lattice distribution F for *i.i.d.* inter-renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ such that $\mathbb{E}X_1^2 < \infty$. For each $x \in \mathbb{R}_+$, we can define an alternating renewal process $W : \Omega \to [0,1]^{\mathbb{R}_+}$ defined as $W_t \triangleq \mathbb{1}_{\{A_t \leqslant x\}}$.

(a) Show that W is a regenerative alternating process.

(b) Show that its nth on and off times are $Z_n \triangleq X_n \wedge x$ and $Y_n \triangleq X_n - Z_n$ respectively.

(c) Repeat the same exercise when on times are excess time being less than a threshold x.

(d) Show that the limiting age and excess time distributions are identical to F_e.

(e) Show that the limiting mean of age and excess times satisfy the following equality,

$$\lim_{t \to \infty} \mathbb{E}Y_t = \lim_{t \to \infty} \mathbb{E}A_t = \frac{\mathbb{E}X_1^2}{2\mathbb{E}X_1}.$$

(f) Show that $\lim_{t\to\infty} \left(m_t - \frac{t}{\mathbb{E}X_1} \right) = \frac{\mathbb{E}X_1^2}{2(\mathbb{E}X_1)^2} - 1$.