Lecture-14: Equilibrium renewal sequences

1 Ergodic theorem

Theorem 1.1 (Ergodicity). *Consider a delayed regenerative process $Z : \Omega \to \mathcal{Z}^{\mathbb{R}_+}$ with delayed regeneration instants S, independent inter renewal times X, and a Borel measurable set $A \in \mathcal{B}(\mathcal{Z})$. If $\mathbb{E}X_1, \mathbb{E}X_2 \in (0,\infty)$, then we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_{t\in\mathbb{R}_+} \mathbb{1}_{\{Z_t\in A\}}dt = \lim_{t\to\infty} \mathbb{E}\mathbb{1}_{\{Z_t\in A\}}.$$

Proof. We denote the delayed renewal function for independent inter renewal times X by m^D, and define probability function $f \in [0,1]^{\mathbb{R}_+}$ and two associated kernel functions $K^1, K^2 \in [0,1]^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$ as

$$f_t \triangleq P\{Z_t \in A\}, \qquad K^1_t \triangleq P\{Z_t \in A, S_1 > t\}, \qquad K^2_t \triangleq P\{Z_{S_1+t} \in A, X_2 > t\}.$$

In this case, we can write the renewal equation as $f = K^1 + K^2 * m^D$ where K^1, K^2 are directly Riemann integrable functions. Hence, it follows from key renewal theorem that

$$\lim_{t\to\infty} \mathbb{E}\mathbb{1}_{\{Z_t\in A\}} = \lim_{t\to\infty} P\{Z_t \in A\} = \frac{\int_{t\in\mathbb{R}_+} K^2_t dt}{\mathbb{E}X_2}.$$

For the renewal sequence S, we define a continuously accruing reward process $Q : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$ as $Q_t \triangleq \int_0^t \mathbb{1}_{\{Z_u\in A\}}du$. The cumulative reward in nth renewal duration $[S_{n-1}, S_n)$ is given by

$$\int_{S_{n-1}}^{S_n} \mathbb{1}_{\{Z_u\in A\}}du = \int_0^{X_n} \mathbb{1}_{\left\{Z_{S_{n-1}+u}\in A\right\}}du = \int_{u\in\mathbb{R}_+} \mathbb{1}_{\left\{Z_{S_{n-1}+u}\in A, X_n>u\right\}}du.$$

From the regenerative property of Z, we have $\mathbb{E}\int_{S_{n-1}}^{S_n} \mathbb{1}_{\{Z_u\in A\}}du = \mathbb{E}\int_{t\in\mathbb{R}_+} K^2_t dt$. Since $\mathbb{E}X_1, \mathbb{E}X_2 < \infty$, it follows from the renewal reward process that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{1}_{\{Z_u\in A\}}du = \lim_{t\to\infty} \frac{1}{t}Q_t = \frac{1}{\mathbb{E}X_2}\mathbb{E}\int_{S_{n-1}}^{S_n} \mathbb{1}_{\{Z_u\in A\}}du = \frac{1}{\mathbb{E}X_2}\mathbb{E}\int_{t\in\mathbb{R}_+} K^2_t dt.$$

□

Example 1.2 (Alternating renewal processes). Recall that an *i.i.d.* nonnegative sequence (Z, Y) is an alternating renewal sequence, with *i.i.d.* inter renewal time sequence X, renewal sequence S, counting process N, and alternating renewal process W defined for any $t \in \mathbb{R}_+$ as $W_t \triangleq \mathbb{1}_{\left\{A_t \leqslant Z_{N_t+1}\right\}}$. If $\mathbb{E}X_1 > 0$, then N_t is almost surely finite at any time t, and we can write

$$W_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{\{N_t=n-1\}}\mathbb{1}_{\left\{A_t\leqslant Z_{N_t+1}\right\}} = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{N_t=n-1\}}\mathbb{1}_{\{t-S_{n-1}\leqslant Z_n\}} = \sum_{n\in\mathbb{N}} \mathbb{1}_{(0,Z_n]}(t - S_{n-1})_+.$$

Recall that W is a regenerative process at the regeneration points being the renewal instants S for inter renewal times X. Further $\mathbb{1}_{\{W_t=1\}} = W_t$ for each $t \in \mathbb{R}_+$, thus it follows from Theorem 1.1 that if $\mathbb{E}X_1 \in (0,\infty)$, then

$$\lim_{t\to\infty} \mathbb{E}W_t = \lim_{t\to\infty} \frac{1}{t}\int_0^t W_u du.$$

Theorem 1.3 (Ergodicity). *Consider a delayed regenerative process $Z : \Omega \to \mathcal{Z}^{\mathbb{R}_+}$ with delayed regeneration instants S, independent inter renewal times X, and a Borel measurable set $A \in \mathcal{B}(\mathcal{Z})$. If $\mathbb{E}X_1, \mathbb{E}X_2 \in (0,\infty)$ and integrals $\int_{t\in\mathbb{R}_+} \mathbb{E}\left|Z_t\right| \mathbb{1}_{\{S_1>t\}}dt, \int_{t\in\mathbb{R}_+} \mathbb{E}\left|Z_{S_1+t}\right| \mathbb{1}_{\{X_2>t\}}dt$ are finite, then we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_{t\in\mathbb{R}_+} Z_t dt = \lim_{t\to\infty} \mathbb{E}Z_t.$$

Proof. We denote the delayed renewal function for independent inter renewal times X by m^D, and define function $f \in \mathbb{R}^{\mathbb{R}_+}$ and two associated kernel functions $K^1, K^2 \in \mathbb{R}^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$ as

$$f_t \triangleq \mathbb{E}Z_t, \qquad K^1_t \triangleq \mathbb{E}Z_t \mathbb{1}_{\{S_1>t\}}, \qquad K^2_t \triangleq \mathbb{E}Z_{S_1+t}\mathbb{1}_{\{X_2>t\}}.$$

In this case, we can write the renewal equation as $f = K^1 + K^2 * m^D$ where K^1, K^2 are directly Riemann integrable functions. Hence, it follows from key renewal theorem that

$$\lim_{t\to\infty} f_t = \lim_{t\to\infty} \mathbb{E}Z_t = \frac{\int_{t\in\mathbb{R}_+} K^2_t dt}{\mathbb{E}X_2}.$$

For the renewal sequence S, we define a continuously accruing reward process $Q : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$ as $Q_t \triangleq \int_0^t Z_u du$. The cumulative reward in nth renewal duration $[S_{n-1}, S_n)$ is given by

$$\int_{S_{n-1}}^{S_n} Z_u du = \int_0^{X_n} Z_{S_{n-1}+u} du = \int_{u\in\mathbb{R}_+} Z_{S_{n-1}+u}\mathbb{1}_{\{X_n>u\}} du.$$

From the regenerative property of Z, we have $\mathbb{E}\int_{S_{n-1}}^{S_n} Z_u du = \mathbb{E}\int_{t\in\mathbb{R}_+} K^2_t dt$. Since $\mathbb{E}X_1, \mathbb{E}X_2 < \infty$, it follows from the renewal reward process that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Z_u du = \lim_{t\to\infty}\frac{1}{t} Q_t = \frac{1}{\mathbb{E}X_2}\mathbb{E}\int_{S_{n-1}}^{S_n} Z_u du = \frac{1}{\mathbb{E}X_2}\mathbb{E}\int_{t\in\mathbb{R}_+} K^2_t dt.$$

□

Example 1.4 (Age and excess times). Consider a delayed renewal sequence S with independent step-size sequence X, associated counting process N, age process A, and excess time process Y. Let G denote the distribution of X_1 and F denote the common distribution of X_n for $n \geqslant 2$. Recall that A and Y are both positive delayed regenerative processes at the regeneration points being the delayed renewal instants S. We further observe that

$$\int_{t\in\mathbb{R}_+} \mathbb{E}A_t \mathbb{1}_{\{S_1>t\}} dt = \int_{t\in\mathbb{R}_+} t\bar{G}(t)dt = \mathbb{E}X_1^2, \quad \int_{t\in\mathbb{R}_+} \mathbb{E}A_{S_1+t}\mathbb{1}_{\{X_2>t\}}dt = \int_{t\in\mathbb{R}_+} t\bar{F}(t)dt = \mathbb{E}X_2^2.$$

Similarly, we can find that $\int_{t\in\mathbb{R}_+} \mathbb{E}Y_t\mathbb{1}_{\{S_1>t\}}dt = \mathbb{E}X_1^2$ and $\int_{t\in\mathbb{R}_+} \mathbb{E}Y_{S_1+t}\mathbb{1}_{\{X_2>t\}}dt = \mathbb{E}X_2^2$. Since $\mathbb{E}X^2 \geqslant (\mathbb{E}X)^2$, it follows that if $\mathbb{E}X_1^2$ and $\mathbb{E}X_2^2$ are finite, then so are $\mathbb{E}X_1, \mathbb{E}X_2$. Thus , it follows from Theorem 1.3 that if $\mathbb{E}X_1^2, \mathbb{E}X_2^2$ are finite and $\mathbb{E}X_1, \mathbb{E}X_2$ are positive, then

$$\lim_{t\to\infty} \mathbb{E}A_t = \lim_{t\to\infty}\frac{1}{t}\int_0^t A_u du, \qquad \lim_{t\to\infty}\mathbb{E}Y_t = \lim_{t\to\infty}\frac{1}{t}\int_0^t Y_u du.$$

Recall that $X_{N_t+1} = A_t + Y_t$ for each $t \in \mathbb{R}_+$ and from the linearity of integration and expectation, we obtain that

$$\lim_{t\to\infty}\mathbb{E}X_{N_t+1} = \lim_{t\to\infty}\frac{1}{t}\int_0^t X_{N_u+1}du = \frac{\mathbb{E}X_1^2}{\mathbb{E}X_1}.$$

Example 1.5 (Renewal reward process). Consider a positive renewal reward process Q for a renewal sequence S with inter renewal times X, associated counting process N, continuous positive accrual of rewards, and *i.i.d.* aggregate reward $R_n \triangleq Q_{S_n} - Q_{S_{n-1}}$ in renewal duration $(S_{n-1}, S_n]$ for each $n \in \mathbb{N}$. Recall that the reward at the end of current renewal interval is a regenerative process with regeneration points S. Applying monotone convergence theorem to exchange integral and mean for positive integrands, we observe that $\int_{t\in\mathbb{R}_+}\mathbb{E}\left|R_{N_t+1}\right|\mathbb{1}_{\{S_1>t\}}dt = \mathbb{E}[R_1X_1]$. From Theorem 1.3, it follows that if $\mathbb{E}[R_1X_1]$ is finite and $\mathbb{E}X_1 \in (0,\infty)$, then

$$\lim_{t\to\infty}\mathbb{E}R_{N_t+1} = \lim_{t\to\infty}\frac{1}{t}\int_0^t R_{N_u+1}du = \frac{\mathbb{E}[R_1X_1]}{\mathbb{E}[X_1]}.$$

Let $g \in \mathbb{R}_+^{\mathbb{R}_+}$ be an increasing function, then we have $\mathbb{E}[X_1 g(X_1)] \geqslant \mathbb{E}X_1\mathbb{E}g(X_1)$ by Chebyshev's inequality. It follows that if $R_n \triangleq g(X_n)$ for each $n \in \mathbb{N}$ for some increasing function g, then it follows that $\lim_{t\to\infty}\mathbb{E}R_{N_t+1} \geqslant \mathbb{E}R_1$.

2 Equilibrium renewal sequence

Definition 2.1 (Equilibrium distribution). Consider a delayed renewal process S with independent inter renewal times X with distribution G for X_1 and common distribution F for X_n where $n \geqslant 2$. The limiting marginal distribution of age process A is called the *equilibrium distribution* of renewal sequence S and is denoted by $F_e \in [0,1]^{\mathbb{R}_+}$ where for each $x \in \mathbb{R}_+$,

$$F_e(x) \triangleq \lim_{t\to\infty} P\{A_t \leqslant x\}.$$

Theorem 2.2 (Limiting marginal distribution of age and excess time). *Consider a delayed renewal process S with finite mean independent inter renewal times X such that the distribution of first renewal time is G, and the distribution of subsequent renewal times are identically F. Denoting the associated counting process by N^D and renewal function m^D, we can find the limiting complementary probability distribution of age for each $x \in \mathbb{R}_+$ as*

$$\bar{F}_e(x) = \lim_{t\to\infty} P\{A_t \geqslant x\} = \frac{1}{\mathbb{E}X_2}\int_x^\infty \bar{F}(y)dy.$$

Proof. Recall that the age process is delayed regenerative at the regeneration points S. Therefore, we can find its marginal distribution at time $t \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ as

$$P\{A_t \geqslant x\} = \mathbb{1}_{\{t\geqslant x\}}\bar{G}_t + \int_0^t dm^D_{t-y}\mathbb{1}_{\{y\geqslant x\}}\bar{F}(y).$$

Since $0 \leqslant \mathbb{1}_{\{y\geqslant x\}}\bar{F}(y) \leqslant \bar{F}(y)$ where $\bar{F}$ is directly Riemann integrable, it follows that $\mathbb{1}_{[0,x]}\bar{F}$ is also directly Riemann integrable. Taking limit $t \to \infty$ on both sides and applying key renewal theorem, we obtain the result. □

Remark 1. Similar to Theorem 2.2, we can show that the limiting complementary marginal distribution of excess time process is $\lim_{t\to\infty} P\{Y_t \geqslant x\} = \bar{F}_e(x)$ for each $x \in \mathbb{R}_+$.

Corollary 2.3. *Consider a delayed renewal process S with finite mean independent inter renewal times X such that the distribution of first renewal time is G, and the distribution of subsequent renewal times are identically F. The moment generating function for limiting age or excess time is $\mathcal{L}_{F_e}(s) = \frac{1-\mathcal{L}_F(s)}{s\mathbb{E}X_2}$ when it exists.*

Proof. By definition of moment generating function, we have $\mathcal{L}_{F_e}(s) = \mathbb{E}\left[e^{-s\lim_{t\to\infty}A_t}\right]$, where the distribution of $\lim_{t\to\infty}A_t$ is F_e. We use integration by parts, to write

$$\mathcal{L}_{F_e}(s) = \int_0^\infty e^{-sx}dF_e(x) = \frac{1}{\mathbb{E}X_2}\int_0^\infty e^{-sx}\bar{F}(x)dx = \frac{1}{s\mathbb{E}X_2} - \frac{1}{s\mathbb{E}X_2}\int_0^\infty e^{-sx}dF(x) = \frac{1}{s\mathbb{E}X_2}(1-\mathcal{L}_F(s)).$$

□

Definition 2.4. A delayed renewal process S with independent inter renewal times X having common distribution F for $(X_n, n \geqslant 2)$ and the initial arrival distribution $G = F_e$ is called the *equilibrium renewal sequence*. The associated age and excess time processes are denoted by A^e and Y^e respectively, and the counting process and renewal functions are denoted by N^e and m^e respectively.

Remark 2. Observe that F_e is the limiting distribution of the age and the excess time for the renewal process with common inter renewal time distribution F. Hence, if we start observing a renewal process at some arbitrarily large time t, then the observed renewal process is the equilibrium renewal sequence. This delayed renewal process exhibits stationary properties. That is, the limiting behaviors are exhibited for all times.

Theorem 2.5 (Renewal function). *The renewal function m^e for the equilibrium renewal sequence is linear in time, i.e. $m^e_t = t/\mathbb{E}X_2$ at any time $t \in \mathbb{R}_+$.*

Proof. Laplace transform of a delayed renewal function is $\mathcal{L}_G(s)/(1-\mathcal{L}_F(s))$ where G and F are distributions of first and second inter renewal times. Further, the Laplace transform of first renewal time in equilibrium renewal sequence is $(1-\mathcal{L}_F(s))/(s\mathbb{E}X_2)$ from Corollary 2.3. Combining the two results, we obtain the Laplace transform of renewal function m_e as $\mathcal{L}_{m^e}(s) = 1/s\mathbb{E}X_2$ for all s in the region of convergence. Further, we know that for any $a \in \mathbb{R}$ the Laplace transform of function at is given by $\mathcal{L}_{at}(s) = a\int_0^\infty e^{-st}dt = a/s$. Since moment generating function is a one-to-one map, the result follows. □

Theorem 2.6 (Age and excess time). *The distribution of age A^e_t and excess time Y^e_t for the equilibrium renewal sequence are stationary. In particular, for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$, we have*

$$P\{A^e_t > x\} = \mathbb{1}_{\{t>x\}}\bar{F}_e(x), \qquad P\{Y^e_t > x\} = \bar{F}_e(x).$$

Proof. Recall that the age process A^e and the excess time process Y^e are delayed regenerative processes with the distribution of first renewal time being F_e, the subsequent inter renewal time distribution F, and renewal function m^e defined as $m^e_t \triangleq t/\mathbb{E}X_2$ for all $x,t \in \mathbb{R}_+$. Thus, their marginal tail probabilities at time $t \in \mathbb{R}_+$ and for $x \in \mathbb{R}_+$ are

$$P\{A^e_t \geqslant x\} = \mathbb{1}_{\{t\geqslant x\}}\bar{F}_e(t) + \frac{1}{\mathbb{E}X_2}\int_x^t \bar{F}(y)dy, \qquad P\{Y^e_t \geqslant x\} = \bar{F}_e(t+x) + \frac{1}{\mathbb{E}X_2}\int_x^{t+x}\bar{F}(y)dy.$$

The result follows from substituting $\bar{F}_e(x) = \int_x^\infty \bar{F}(y)dy/\mathbb{E}X_2$ in the above equation. □

Remark 3. When we start observing the equilibrium counting process N^e at time s, the observed renewal process is delayed renewal process with initial distribution Y^e at time s being identical to the distribution F_e. Hence, the number of renewals $N^e_{t+s} - N^e_s$ has the same distribution as N^e_t in duration $(0,t]$. That is, the distribution of counting process is shift invariant.

Theorem 2.7 (Stationary increments). *The counting process N^e for the equilibrium renewal sequence has stationary increments.*

Proof. We can write the event $\{N^e_{s+t} - N^e_s = n\} = \{S_{N^e_s+n} \leqslant t < S_{N^e_s+n+1}\}$ where $S_{N^e_s+n} = Y^e_s + \sum_{k=2}^n X_{N^e_s+k}$. Since Y^e_s is distributed identically to X_1, to show the result it suffices to show that $(X_{N^e_s+k} : k \geqslant 2)$ is *i.i.d.* with common distribution F and independent of Y^e_s. To this end, we consider the function $f \in [0,1]^{\mathbb{R}_+}$ and associated kernel functions $k^1, k^2 \in [0,1]^{\mathbb{R}_+}$, defined for each $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^k_+, y \in \mathbb{R}_+$ as $f_t \triangleq P\Big(\{Y^e_t > y\} \cap_{k=2}^n \Big\{X_{N^e_t+k} \leqslant x_k\Big\}\Big)$, $k^1_t \triangleq P\Big(\{Y^e_t > y, S_1 > t\} \cap_{k=2}^n \Big\{X_{N^e_t+k} \leqslant x_k\Big\}\Big) = \bar{F}_e(t+y)\prod_{k=2}^n F(x_k)$, and $k^2_t \triangleq P\Big(\Big\{Y^e_{S_1+t} > y, t < X_2\Big\} \cap_{k=2}^n \Big\{X_{N^e_{S_1+t}+k} \leqslant x_k\Big\}\Big) = \bar{F}(t+y)\prod_{k=2}^n F(x_k)$. Since excess time process is delayed regenerative, we obtain the renewal function $f = k^1 + k^2 * m^e$. From the definition of equilibrium distribution F_e and equilibrium renewal function m^e, we obtain for each $t, y \in \mathbb{R}_+$ and $x \in \mathbb{R}^k_+$ that

$$f_t = P\Big(\{Y^e_t > y\} \cap_{k=2}^n \Big\{X_{N^e_t+k} \leqslant x_k\Big\}\Big) = \bar{F}_e(y)\prod_{k=2}^n F(x_k).$$

□

Example 2.8 (Poisson process). Consider the case, when inter renewal time distribution F for a delayed renewal sequence is exponential with rate λ. Here, one would expect the equilibrium distribution $F_e = F$, since Poisson process has stationary and independent increments. We observe that for any $x \in \mathbb{R}_+$, we have

$$F_e(x) = \frac{1}{\mu}\int_0^x \bar{F}(y)dy = \lambda\int_0^x e^{-\lambda y}dy = 1 - e^{-\lambda x} = F(x).$$

We see that F_e is also distributed exponentially with rate λ. Indeed, this is a Poisson process with rate λ.

Example 2.9 (Age and excess time process as an alternating renewal process). Consider an aperiodic renewal sequence with distribution F for *i.i.d.* inter renewal times X such that $\mathbb{E}X_1^2 < \infty$. For a fixed $x, y \in \mathbb{R}_+$, we can define alternating renewal processes $W, V : \Omega \to [0,1]^{\mathbb{R}_+}$ defined as $W_t \triangleq \mathbb{1}_{\{A_t \leqslant x\}}$ and $V_t \triangleq \mathbb{1}_{\{Y_t > y\}}$ for all $t \in \mathbb{R}_+$. We observe that the nth on and off times are $X_n \wedge x$ and $(X_n - x)_+$ respectively for the process W, and $(X_n - y)_+$ and $X_n \wedge y$ for the process V. We can consider two reward processes Q_W and Q_V defined as $\int_0^t W_u du$ and $\int_0^t V_u du$ for each $t \in \mathbb{R}_+$. From the renewal reward theorem, we obtain that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t W_u du = \frac{\mathbb{E}X_1 \wedge x}{\mathbb{E}X_1} = F_e(x), \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t V_u du = \frac{\mathbb{E}(X_1 - x)_+}{\mathbb{E}X_1} = \bar{F}_e(y).$$

Since the mean of the distribution F_e is $\mathbb{E}X_1^2/2\mathbb{E}X_1$, we obtain the following equality

$$\lim_{t\to\infty} \mathbb{E}Y_t = \lim_{t\to\infty} \frac{1}{t}\int_0^t Y_u du = \lim_{t\to\infty} \mathbb{E}A_t = \lim_{t\to\infty} \frac{1}{t}\int_0^t A_u du = \frac{\mathbb{E}X_1^2}{2\mathbb{E}X_1}.$$