# Lecture-16: Class Properties

## 1 Communicating classes

**Definition 1.1.** Let $x, y \in \mathcal{X}$. If $p_{xy}^{(n)} > 0$ for some $n \in \mathbb{Z}_+$, then we say that state $y$ is *accessible* from state $x$ and denote it by $x \to y$. If two states $x, y \in \mathcal{X}$ are accessible to each other, they are said to *communicate* with each other and denoted by $x \leftrightarrow y$. A set of states that communicate are called a *communicating class*.

**Definition 1.2.** A relation $R$ on a set $\mathcal{X}$ is a subset of $\mathcal{X} \times \mathcal{X}$.

**Definition 1.3.** An equivalence relation $R \subseteq \mathcal{X} \times \mathcal{X}$ has following three properties.
Reflexivity: If $x \in \mathcal{X}$, then $(x, x) \in R$.
Symmetry: If $(x, y) \in R$, then $(y, x) \in R$.
Transitivity: If $(x, y), (y, z) \in R$, then $(x, z) \in R$.

*Remark* 1. Equivalence relations partition a set $\mathcal{X}$.

**Proposition 1.4.** *Communication is an equivalence relation.*

*Proof.* Reflexivity follows from zero-step transition, and symmetry follows from the definition of communicating class. For transitivity, it suffices to show that, if $x \to y$ and $y \to z$ then $x \to z$. Since $x \to y$ and $y \to z$, there exists $m, n \in \mathbb{N}$ such that $p_{xy}^{(m)} > 0$ and $p_{yz}^{(n)} > 0$. From Chapman Kolmogorov equations, we have $m + n \in \mathbb{N}$ such that $p_{xz}^{(m+n)} = \sum_{w \in \mathbb{Z}_+} p_{xw}^{(m)} p_{wz}^{(n)} \geqslant p_{xy}^{(m)} p_{yz}^{(n)} > 0$. That is, $x \to z$. □

### 1.1 Irreducibility and periodicity

A consequence of the previous result is that communicating classes are disjoint or identical.

**Definition 1.5.** A Markov chain with a single communicating class is called *irreducible*.

**Definition 1.6.** A *class property* is the one that is satisfied by all states in the communicating class.

*Remark* 2. We will see many examples of class properties. Once we have shown that a property is a class property, then one only needs to check that one of the states in the communicating class has the property for the entire class to have that.

**Definition 1.7.** We denote the set of possible recurrence times for a Markov chain with transition probability matrix $p \in \mathcal{M}(\mathcal{X})^{\mathcal{X}}$ as $A_x \triangleq \left\{ n \in \mathbb{N} : p_{xx}^{(n)} > 0 \right\}$.

*Remark* 3. If one can revisit a state $x$ in $m$ and $n$ steps, then also in $m + n$ steps, since $p_{xx}^{(m+n)} \geqslant p_{xx}^{(m)} p_{xx}^{(n)}$. It follows that set $A_x$ is closed under addition for all $x \in \mathcal{X}$.

**Definition 1.8.** The *period* of state $x$ is defined as $d(x) \triangleq \gcd(A_x)$. If the period is 1, we say the state is *aperiodic*.

**Proposition 1.9.** *Periodicity is a class property.*

*Proof.* We will show that for two communicating states $x \leftrightarrow y$, the periodicities are identical. We will show that $d(x) | d(y)$ and $d(y) | d(x)$. We choose $m, n \in \mathbb{N}$ such that

$$p_{xx}^{(m+n)} \geqslant p_{xy}^{(m)} p_{yx}^{(n)} > 0, \qquad p_{yy}^{(m+n)} \geqslant p_{yx}^{(n)} p_{xy}^{(m)} > 0.$$

It follows that $m + n \in A_x \cap A_y$. Let $s \in A_x$, then it follows that $m + n + s \in A_y$, since $p_{yy}^{(n+s+m)} \geqslant p_{yx}^{(n)} p_{xx}^{(s)} p_{xy}^{(m)} > 0$. Hence $d(y) | n + m$ and $d(y) | n + s + m$ which implies $d(y) | s$. Since the choice of $s \in A_x$ was arbitrary, it follows that $d(y) | d(x)$. Similarly, we can show that $d(x) | d(y)$. □

**Example 1.10 (Random walk on a ring).** Let $G = (\mathcal{X}, E)$ be a finite graph where $\mathcal{X} \triangleq \{0, \dots, n-1\}$ and $E = \{(x, x+1) : x \in \mathcal{X}\}$ where addition is modulo $n$. Let $Z : \Omega \to \{-1, 1\}^{\mathbb{N}}$ be a random *i.i.d.* sequence of step-sizes with $\mathbb{E}Z_n = 2p - 1$. We denote the location of particle after $n$ random steps by $X_n \triangleq X_0 + \sum_{i=1}^{n} Z_i$. It follows that the random walk $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ is an irreducible homogeneous Markov chain with period 2 if $n$ is even. The Markov chain $X$ is aperiodic if $n$ is odd.

**Proposition 1.11.** *Consider a Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ on a finite state space $\mathcal{X}$. If $X$ is irreducible and aperiodic, then there exists $n_0 \in \mathbb{N}$ such that $p_{xy}^{(n)} > 0$ for all $x, y \in \mathcal{X}$ and $n \geqslant n_0$.*

*Proof.* We fix a pair of states $x, y \in \mathcal{X}$. Since periodicity is a class property and $X$ is aperiodic, it follows that $\gcd(A_x) = 1$. From Lemma A.2, there exists $m_x \in A_x$ such that $n \in A_x$ for all $n \geqslant m_x$. From the irreducibility of Markov chain $X$, we can find $n_{xy} \in \mathbb{N}$ such that $p_{xy}^{(n_{xy})} > 0$. It follows that $p_{xy}^{(n)} > 0$ for all $n \geqslant m_x + n_{xy} \in \mathbb{N}$. Since the state space $\mathcal{X}$ is finite, we have a finite $n_0 \triangleq \sup_{x \in \mathcal{X}} m_x + \sup_{x,y \in \mathcal{X}} n_{xy} \in \mathbb{N}$ such that $p_{xy}^{(n)} > 0$ for any state $x, y \in \mathcal{X}$ for all $n \geqslant n_0$. □

# 2 Transient and recurrent states

## 2.1 Hitting and return times

**Definition 2.1.** Consider a homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$. Let state $x \in \mathcal{X}$. We define $\tau_x^+(0) \triangleq 0$ and inductively define the *kth hitting time* to state $x$, as

$$\tau_x^+(k) \triangleq \inf\{n > \tau_x^+(k-1) : X_n = x\}.$$

If $X_0 = x$, then $\tau_x^+(1)$ is called the *first return time* to state $x$.

**Lemma 2.2.** *Consider a Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ on finite state space $\mathcal{X}$. If $X$ is irreducible, then $\mathbb{E}_x \tau_y^+(1) < \infty$ for all states $x, y \in \mathcal{X}$.*

*Proof.* From the definition of irreducibility, for each pair of states $z, w \in \mathcal{X}$, we have a positive integer $n_{zw} \in \mathbb{N}$ such that $p_{zw}^{n_{zw}} > \epsilon_{zw} > 0$. Since the state space $\mathcal{X}$ is finite, we define

$$\epsilon \triangleq \inf_{z,w \in \mathcal{X}} \epsilon_{zw} > 0, \qquad r \triangleq \sup_{z,w \in \mathcal{X}} n_{zw} \in \mathbb{N}.$$

Hence, there exists a positive integer $r \in \mathbb{N}$ and a real $\epsilon > 0$ such that $p_{zy}^{(n)} > \epsilon$ for some $n \leqslant r$ and all states $z, y \in \mathcal{X}$. It follows that $P_z(\cup_{n \in [r]} \{X_n = y\}) > \epsilon$ or $P_z\left\{\tau_y^+(1) > r\right\} < 1 - \epsilon$ for any initial condition $X_0 = z \in \mathcal{X}$ and state $y \in \mathcal{X}$. Fix $k \in \mathbb{N}$. We observe that $\left\{\tau_y^+(1) > kr\right\} = \cup_{z \neq y}\left\{\tau_y^+(1) > kr, \tau_y^+(1) > (k-1)r, X_{(k-1)r} = z\right\}$. Therefore,

$$P_x\left\{\tau_y^+(1) > kr\right\} = \sum_{z \neq y} P_x\left\{\tau_y^+(1) > (k-1)r, X_{(k-1)r} = z\right\} P\Big(\left\{\tau_y^+(1) > kr\right\} \mid \left\{X_{(k-1)r} = z, \tau_y^+(1) > (k-1)r, X_0 = x\right\}\Big).$$

We observe that $\left\{X_{(k-1)r} = z, \tau_y^+(1) > (k-1)r, X_0 = x\right\} \in \mathcal{F}_{(k-1)r}$ for all $z \neq y$. From the Markov property and the time homogeneity of $X$, we can write

$$\begin{aligned} P(\left\{\tau_y^+(1) > kr\right\} \mid \left\{X_{(k-1)r} = z, \tau_y^+(1) > (k-1)r, X_0 = x\right\}) &= P(\left\{\tau_y^+(1) > kr\right\} \mid \left\{X_{(k-1)r} = z\right\}) \\ &= P_z\left\{\tau_y^+(1) > r\right\} < (1 - \epsilon). \end{aligned}$$

It follows that $P_x\left\{\tau_y^+(1) > kr\right\} < P_x\left\{\tau_y^+(1) > (k-1)r\right\}(1 - \epsilon)$. By induction, we have $P_x\left\{\tau_y^+(1) > kr\right\} < (1 - \epsilon)^k$. Since $P_x\left\{\tau_y^+(1) > n\right\}$ is decreasing in $n$, we can write

$$\mathbb{E}_x \tau_y^+(1) = \sum_{k \in \mathbb{Z}_+} \sum_{i=0}^{r-1} P_x\left\{\tau_y^+(1) > kr + i\right\} \leqslant \sum_{k \in \mathbb{Z}_+} r P_x\left\{\tau_y^+(1) > kr\right\} < \frac{r}{\epsilon} < \infty.$$

□

**Corollary 2.3.** *Consider a Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ on finite state space $\mathcal{X}$. If $X$ is irreducible, then $P_x\left\{\tau_y^+(1) < \infty\right\} = 1$ for all states $x, y \in \mathcal{X}$.*

*Proof.* This follows from the fact that $\tau_y^+(1)$ is a positive random variable with finite mean for all states $y \in \mathcal{X}$ and any initial state $x \in \mathcal{X}$. □

## 2.2 Recurrence and transience

**Definition 2.4.** We denote the probability of the first transition into state $y$ at time $n$ from the initial state $x$ by $f_{xy}^{(n)} \triangleq P_x\left\{\tau_y^+(1) = n\right\}$. The probability of eventually entering state $y$ from the initial state $x$ is denoted by $f_{xy} \triangleq P_x\left\{\tau_y^+(1) < \infty\right\} = \sum_{n=1}^{\infty} f_{xy}^{(n)}$.

**Definition 2.5.** A state $y$ is said to be *transient* if $f_{yy} < 1$, *recurrent* if $f_{yy} = 1$, and *positive recurrent* if $\mathbb{E}_y \tau_y^+ < \infty$.

**Definition 2.6.** For a discrete time process $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, the total number of visits to a state $y \in \mathcal{X}$ in first $n$ steps is denoted by $N_y(n) \triangleq \sum_{i=1}^{n} \mathbb{1}_{\{X_i = y\}}$. The total number of visits to state $y \in \mathcal{X}$ is denoted by $N_y(\infty)$.

*Remark* 4. From the linearity of expectations and monotone convergence theorem, we get $\mathbb{E}_y N_y(\infty) = \sum_{n \in \mathbb{N}} p_{yy}^{(n)}$.

**Lemma 2.7.** *Consider a time homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$. For each $m \in \mathbb{Z}_+$ and state $x, y \in \mathcal{X}$, we have*

$$P_x\left\{N_y(\infty) = m\right\} = \begin{cases} 1 - f_{xy} & m = 0, \\ f_{xy} f_{yy}^{m-1}(1 - f_{yy}) & m \in \mathbb{N}. \end{cases}$$

*Proof.* We can write the following equality

$$\left\{N_y(\infty) = m\right\} = \left\{\tau_y^+(m) < \infty, \tau_y^+(m+1) = \infty\right\} = \cap_{k=1}^{m} \left\{\tau_y^+(k) - \tau_y^+(k-1) < \infty\right\} \cap \left\{\tau_y^+(m+1) - \tau_y^+(m) = \infty\right\}.$$

For each $k \in \mathbb{N}$, the $k$th hitting time $\tau_y^+(k)$ to the state $y$ is adapted to the natural filtration $\mathcal{F}_\bullet$ of the process $X$. From strong Markov property, the next return to state $y$ is independent of the past. That is, $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ is a delayed renewal sequence for initial state $X_0 = x \neq y$. It follows that

$$P_x\left\{N_y(\infty) = m\right\} = P_x\left\{\tau_y^+(1) < \infty\right\} \prod_{k=2}^{m} P_y\left\{\tau_y^+(k) - \tau_y^+(k-1) < \infty\right\} P_y\left\{\tau_y^+(m+1) = \infty\right\}.$$

□

**Corollary 2.8.** *For a homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, we have*

$$P_y\left\{N_y(\infty) < \infty\right\} = \mathbb{1}_{\{f_{yy} < 1\}}, \qquad \mathbb{E}_y N_y(\infty) = \frac{f_{yy}}{1 - f_{yy}}.$$

*Proof.* From the additivity of probability of disjoint events and the expression for the conditional probability mass function $P_y\left\{N_y(\infty) = m\right\}$ in Lemma 2.7, we obtain

$$P_y\left\{N_y(\infty) < \infty\right\} = \sum_{m \in \mathbb{Z}_+} P_y\left\{N_y(\infty) = m\right\} = (1 - f_{yy}) \sum_{m \in \mathbb{Z}_+} f_{yy}^m = \mathbb{1}_{\{f_{yy} < 1\}}.$$

Similarly, we obtain the mean $\mathbb{E}_y N_y(\infty)$ using the conditional distribution of $N_y(\infty)$ given initial state $y$. □

*Remark* 5. In particular, this corollary implies the following.

1. A transient state is visited a finite amount of times almost surely.
2. A recurrent state is visited infinitely often almost surely.
3. Since $\sum_{y \in \mathcal{X}} N_y(\infty) = \infty$, it follows that not all states can be transient in a finite state Markov chain.

**Proposition 2.9.** *Consider a time homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with probability transition matrix $p \in \mathcal{M}(\mathcal{X})^{\mathcal{X}}$. A state $y \in \mathcal{X}$ is recurrent iff $\sum_{k \in \mathbb{N}} p_{yy}^{(k)} = \infty$.*

*Proof.* For any state $y \in \mathcal{X}$, we can write $p_{yy}^{(k)} = P_y\{X_k = y\} = \mathbb{E}_y \mathbb{1}_{\{X_k=y\}}$. Using monotone convergence theorem to exchange expectation and summation, we obtain $\sum_{k\in\mathbb{N}} p_{yy}^{(k)} = \mathbb{E}_y \sum_{k\in\mathbb{N}} \mathbb{1}_{\{X_k=y\}} = \mathbb{E}_y N_y(\infty)$. Thus, $\sum_{k\in\mathbb{N}} p_{yy}^{(k)}$ represents the expected number of returns $\mathbb{E}_y N_y(\infty)$ to a state $y$ starting from state $y$, which we know to be finite if the state is transient and infinite if the state is recurrent. □

**Proposition 2.10.** *Transience and recurrence are class properties.*

*Proof.* Let $x \leftrightarrow y$. Then from the reachability, there exist some $m, n > 0$, such that $p_{xy}^{(m)} > 0$ and $p_{yx}^{(n)} > 0$. Let $x$ be a recurrent state, then $\sum_{s\in\mathbb{Z}_+} p_{xx}^{(s)} = \infty$. We show that $\sum_{k\in\mathbb{Z}_+} p_{yy}^{(k)} = \infty$, by observing

$$\sum_{k\in\mathbb{Z}_+} p_{yy}^{(k)} \geqslant \sum_{s\in\mathbb{Z}_+} p_{yy}^{(m+n+s)} \geqslant \sum_{s\in\mathbb{Z}_+} p_{yx}^{(n)} p_{xx}^{(s)} p_{xy}^{(m)} = \infty.$$

Let $x$ be a transient state, then $\sum_{s\in\mathbb{Z}_+} p_{xx}^{(s)} < \infty$. We show that $\sum_{k\in\mathbb{Z}_+} p_{yy}^{(k)} < \infty$, by observing

$$\sum_{k\in\mathbb{Z}_+} p_{yy}^{(k)} \leqslant \frac{\sum_{k\in\mathbb{Z}_+} p_{xx}^{(m+n+k)}}{p_{xy}^{(m)} p_{yx}^{(n)}} \leqslant \frac{\sum_{s\in\mathbb{Z}_+} p_{xx}^{(s)}}{p_{xy}^{(m)} p_{yx}^{(n)}} < \infty.$$

□

**Exercise 2.11.** If $y$ is recurrent and $x$ such that $x \leftrightarrow y$, then show that $f_{xy} = 1$.

# A Bézout's identity

**Lemma A.1 (Bézout).** *Consider a vector $a \in (\mathbb{Z} \setminus \{0\})^n$ with gcd $d$, and define the set $S \triangleq \{\langle a, x\rangle : x \in \mathbb{Z}^n, \langle a, x\rangle > 0\}$. Then $d$ is the smallest element of set $S$ and $d|s$ for all $s \in S$.*

*Proof.* Let $I \triangleq \{i \in [n] : a_i > 0\}$, then $x \in \{-1, 1\}^n$ defined as $x_i = 1$ for $i \in I$ and $x_i = -1$ for $i \notin I$ ensures that $\langle a, x\rangle = \sum_{i=1}^n |a|_i > 0$. It follows that $S$ is non-empty. Let $g = \langle a, x\rangle$ be the minimum element of $S$.

First, we show that $g|a_i$ for all $i \in [n]$. Let $0 < r_i < g$ be the reminder when $g$ divides $a_i$. Then, we can write $r_i = a_i - g v_i$ for some $v_i \in \mathbb{Z}_+$. Therefore, $r_i = \sum_{j\neq i} -x_j v_i a_j + (1 - v_i x_i) a_i \in S$. However, this is a contraction since $g$ is the smallest element of $S$, and the result follows. Since any $s \in S$ can be written as $s = \langle a, z\rangle$ and $d|a_i$ for all $i \in [n]$, it follows that $d|s$.

Second, we show that if any $c \in \mathbb{Z}_+$ such that $c|a_i$ for all $i \in [n]$, then $c|g$ and hence $g = d$. Since $g = \langle a, x\rangle$, this implies that $c|g$ and the result follows.

□

**Lemma A.2.** *If $A$ is a set closed under addition and* $\gcd(A) = 1$*, then there exists $m_0 \in A$ such that $m \in A$ for all $m \geqslant m_0$.*

*Proof.* Let $A$ be a set generated by positive integers $a \triangleq (a_1, a_2, \dots, a_n)$ such that $A = \left\{\langle a, x\rangle : x \in \mathbb{Z}_+^{\mathbb{N}}, x \neq 0\right\}$. Let $g$ be the smallest element in $A$. If $g = 1$, then $A = \mathbb{N}$ and there is nothing to prove. We consider the case when $g > 1$. From Bézout's Lemma, there exists $v \in \mathbb{Z}^n$ such that $1 = \langle a, v\rangle$. Since $1 \notin A$, it implies that there exists a non-empty subset $I \triangleq \{i \in [n] : v_i < 0\}$. We define

$$c_1 \triangleq \sum_{i\notin I} v_i a_i \in A, \qquad c_2 \triangleq \sum_{i\in I} -v_i a_i \in A.$$

We observe that $c_1 - c_2 = 1$. Since $1 \notin A$, both $c_1, c_2 > 1$. Let $m_0 = c_2^2$, then for any $m \geqslant m_0$, we can write $m = kc_2 + \ell$ where the remainder $0 \leqslant \ell < c_2$. We can write $c_2^2 \leqslant m = kc_2 + \ell < (k+1)c_2$. Since $c_2 > 0$, this implies that $k > c_2 - 1 \geqslant \ell$. Thus, we can write

$$m = kc_2 + \ell = kc_2 + \ell(c_1 - c_2) = (k - \ell)c_2 + \ell c_1 \in A.$$

□