# Lecture-19: Embedded Markov chain and sojourn times

## 1 State Evolution

For a time homogeneous Markov process $X : \Omega \to \mathcal{X}_+^{\mathbb{R}}$ on countable state space $\mathcal{X} \subseteq \mathbb{R}$ with sample paths that are right continuous with left limits (rcll), we wish to characterize the transition probability kernel $P : \mathbb{R}_+ \to \mathcal{M}(\mathcal{X})^{\mathcal{X}}$, where $P_{xy}(t) \triangleq P(\{X_{s+t} = y\} \mid \{X_s = x\})$ for all $s, t \in \mathbb{R}_+$. To this end, we define the sojourn time in any state, the jump times, and the jump transition probabilities.

### 1.1 Jump time and embedded chain

Consider a right continuous countable state stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ and denote its natural filtration by $\mathcal{F}_\bullet \triangleq (\mathcal{F}_t : t \in \mathbb{R}_+)$ where $\mathcal{F}_t \triangleq \sigma(X_s, s \leqslant t)$ for all $t \in \mathbb{R}_+$.

**Definition 1.1 (Jump instants).** Let $S_0 \triangleq 0$, then the $n$th *jump instant* of $X$ is defined inductively as

$$S_n \triangleq \inf\left\{t > S_{n-1} : X_t \neq X_{S_{n-1}}\right\}.$$

The non decreasing random sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is called the *jump instants sequence*.

**Definition 1.2 (Embedded chain).** Sampling the rcll process $X$ at the jump instants in $S$, we derive the associated *embedded jump chain*$Z : \Omega \to \mathcal{X}^{\mathbb{N}}$, such that the state of the process $X$ at the $n$th jump instant $S_n$ is denoted by $Z_n \triangleq X_{S_n}$, for each $n \in \mathbb{N}$.

**Definition 1.3 (Sojourn times and jump counting process).** By definition, the process $X$ remains in state $Z_{n-1}$ during the interval $[S_{n-1}, S_n)$, and the *sojourn time* for the process $X$ in the state $Z_{n-1}$ is defined as $T_n \triangleq S_n - S_{n-1}$. The counting process associated with jump instants sequence $S$ is denoted by $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$, where the number of jumps for process $X$ in duration $(0, t]$ is denoted by

$$N_t \triangleq \sum_{n \in \mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}.$$

**Lemma 1.4.** *Consider a rcll countable state stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with its natural filtration $\mathcal{F}_\bullet$. Each term of the associated jump time sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is adapted to $\mathcal{F}_\bullet$.*

*Proof.* Since $X$ is rcll, it is progressively measurable, and hence the event $\{S_n \leqslant t\} \in \mathcal{F}_t$. □

*Remark* 1. From the definition of jump instants, it follows that the history until time $t$ is

$$\mathcal{F}_t = \sigma(S_0, (Z_0, T_1), (Z_1, T_2), \dots, (Z_{N_t}, A_t)).$$

We can verify that $\mathcal{F}_{S_n} = \sigma(S_0, (Z_0, T_1), \dots, (Z_{n-1}, T_n), Z_n)$.

**Definition 1.5 (Excess time).** From the definition of excess time as the time until next transition, we can write the excess time for rcll process $X$ in state $Z_{N_t}$ at time $t \in \mathbb{R}_+$ as

$$Y_t \triangleq \inf\{s > 0 : X_{t+s} \neq X_t\}.$$

*Remark* 2. We observe that $Y_t$ is the excess remaining time the process spends in state $X_t$ at instant $t$. That is, $X_{t+Y_t} \neq X_t$, and further $T_n = Y_{S_{n-1}}$ for each $n \in \mathbb{N}$.

## 1.2 Holding time in a state for continuous time Markov chains

*Remark* 3. For a time homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$, the distribution of excess time $Y_t$ conditioned on the current state $X_t$, doesn't depend on time $t$. Hence, we can define the following conditional complementary distribution of excess time as

$$\bar{F}_x(u) \triangleq P(\{Y_t > u\} \mid \{X_t = x\}) = P(\{Y_0 > u\} \mid \{X_0 = x\}) = P_x\{Y_0 > u\}.$$

**Lemma 1.6.** *For a homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$, there exists a positive sequence $\nu \in \mathbb{R}_+^{\mathcal{X}}$, such that $\bar{F}_x(u) = e^{-u\nu_x}$ for each state $x \in \mathcal{X}$ and time $t \in \mathbb{R}_+$.*

*Proof.* We fix a state $x \in \mathcal{X}$ and time $t \in \mathbb{R}_+$, and observe that the function $\bar{F}_x \in [0,1]$ is non negative, non increasing, and right continuous in $u$. Using the Markov property and time homogeneity of process $X$, we can show that $\bar{F}_x$ satisfies the semigroup property. In particular,

$$\bar{F}_x(u+v) = P(\{Y_t > u+v\} \mid \{X_t = x\}) = P(\{Y_t > u, Y_{t+u} > v\} \mid \{X_t = x\}) = \bar{F}_x(u)\bar{F}_x(v).$$

The only continuous non increasing function $\bar{F}_x \in [0,1]^{\mathbb{R}_+}$ that satisfies semigroup property is an exponential function with a negative exponent. □

**Example 1.7 (Poisson process).** Consider the Poisson counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with homogeneous rate $\lambda$. Using the stationary independent increment property, we have for all $u \geqslant 0$

$$\bar{F}_i(u) = P(\{Y_t > u\} \mid \{N_t = i\}) = P(\{N_{t+u} = i\} \mid \{N_t = i\}) = P\{N_{t+u} - N_t = 0\} = P\{Y_t > u\} = e^{-\lambda u}.$$

**Lemma 1.8.** *Consider a homogeneous continuous time Markov chain $X$ with jump instant sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. If $S_{n-1}$ is almost surely finite, then sojourn time $T_n$ is a continuous memoryless random variable, and the sequence of sojourn times $(T_j : j \geqslant n)$ is independent of the past $\mathcal{F}_{S_{n-1}}$ conditioned on state $Z_{n-1}$.*

*Proof.* We observe that the sojourn time $T_n$ equals the excess time $Y_{S_{n-1}}$, where the process remains in state $Z_{n-1} = X_{S_{n-1}}$ in the duration $S_{n-1} + [0, T_n)$. Using the strong Markov property at stopping time $S_{n-1}$, we can write the conditional complementary distribution of $T_n$ given any historical event $H \in \mathcal{F}_{S_{n-1}}$ and $u \geqslant 0$ as

$$P(\{T_n > u\} \mid \{Z_{n-1} = x\} \cap H) = P(\{Y_{S_{n-1}} > u\} \mid \{X_{S_{n-1}} = x\} \cap H) = \bar{F}_x(u).$$

□

**Corollary 1.9.** *If $Z_{n-1} = x$, then the holding time $T_n$ is an exponential random variable with rate $\nu_x$.*

**Definition 1.10 (Transition rate).** For a time homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$, the random holding time in a state $x \in \mathcal{X}$ is exponentially distributed with the rate called the *transition rate* out of state $x$ denoted by $\nu_x$.

**Definition 1.11 (Pure jump).** Consider a time homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$. A state $x \in \mathcal{X}$ is called *instantaneous* if $\nu_x = \infty$, *stable* if $\nu_x \in (0, \infty)$, and *absorbing* if $\nu_x = 0$. If $X$ has no instantaneous states, then it is called *pure jump*.

*Remark* 4. Transition rate out of a state $x$ is the inverse of mean holding time in this state $x$, i.e. $\nu_x = 1/\mathbb{E}_x T_1$. Therefore, the mean holding time $\mathbb{E}_x T_1$ in state $x$ is $\infty$ in an absorbing state, zero in an instantaneous state, and almost surely finite and non-zero in a stable state.

**Definition 1.12.** A pure jump continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with

(i) all stable states and $\inf_{x \in \mathcal{X}} \nu_x \geqslant \nu > 0$ is called *stable*, and

(ii) $\sup_{x \in \mathcal{X}} \nu_x \leqslant \nu < \infty$ is called *regular*.

**Example 1.13 (Non-regular continuous time Markov chain).** For the countable state space $\mathbb{N}$, consider the probability transition matrix $P$ such that $p_{n,n+1} = 1$ and the exponential holding times with rate $\nu_n = n^2$ for each state $n \in \mathbb{N}$. Clearly, $\sup_{n \in \mathbb{N}} \nu_n = \infty$, and hence it is not regular.

*Remark* 5. Pure jump continuous time Markov chain with finite stable states are stable and regular. We will focus on pure jump homogeneous continuous time Markov chain over countably infinite states, that are stable and regular.

## 1.3 Jump instants and embedded chain for continuous time Markov chains

**Proposition 1.14.** *For a stable continuous time Markov chain $X$, the jump instants are stopping times.*

*Proof.* For a stable continuous time Markov chain $X$, we let $0 < \nu \leqslant \inf_{x \in \mathcal{X}} \nu_x$. Then, by coupling in Appendix B, we have a sequence of *i.i.d.* random variables $\overline{T} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, such that $T_n \leqslant \overline{T}_n$ almost surely and $\mathbb{E}\overline{T}_n = 1/\nu$ for each $n \in \mathbb{N}$. Defining $\overline{S}_n \triangleq \sum_{i=1}^n \overline{T}_i$, it follows that $S_n \leqslant \overline{S}_n$ for each $n \in \mathbb{N}$. Since $\overline{S}_n$ is sum of $n$ almost surely finite independent random variables, it is finite almost surely. It follows that $S_n$ is finite almost surely. □

**Proposition 1.15.** *For a regular continuous time Markov chain $X$, the number of jumps $N_t$ is almost surely finite in duration $(0,t]$ for all finite times $t \in \mathbb{R}_+$.*

*Proof.* Let $X$ be a regular continuous time Markov chain and $\sup_{x \in \mathcal{X}} \nu_x \leqslant \nu < \infty$. Then, by coupling in Appendix B, we have a sequence of *i.i.d.* random variables $\underline{T} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, such that $T_n \geqslant \underline{T}_n$ almost surely and $\mathbb{E}\underline{T}_n = 1/\nu$ for each $n \in \mathbb{N}$. Defining $\underline{S}_n \triangleq \sum_{i=1}^n \underline{T}_i$ and $\underline{N}_t \triangleq \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\underline{S}_n \leqslant t\}}$, it follows that $S_n \geqslant \underline{S}_n$ for each $n \in \mathbb{N}$ and $N_t \leqslant \underline{N}_t$ for all $t \in \mathbb{R}_+$. Since $\underline{N}$ is a Poisson counting process with finite rate $\nu$, it is almost surely finite at all $t \in \mathbb{R}_+$ and the result follows. □

*Remark* 6. From the strong Markov property and the time-homogeneity of the continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ and definition of embedded chain, we see that for any state $x, y \in \mathcal{X}$

$$P(\{Z_n = y\} \mid \{Z_{n-1} = x\}) = P_{xy}(S_{n-1}, S_n) = P_{xy}(0, T_n).$$

*Remark* 7. From the law of total probability, it follows that for any rcll stochastic process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with countable state space $\mathcal{X}$, the sum of jump transition probabilities $\sum_{y \neq x} P_{xy}(S_{n-1}, S_n) = 1$ for all states $Z_{n-1} = x \in \mathcal{X}$.

**Lemma 1.16.** *For a stable continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$, the jump probability from state $Z_{n-1}$ to state $Z_n$ depends solely on $Z_{n-1}$ and is independent of the the past and the sojourn time $T_n$.*

*Proof.* Since $X$ is stable, each jump instant is a stopping time. Fix states $x, y \in \mathcal{X}$ and a historical event $H \in \mathcal{F}_{S_{n-1}}$. From the definition of conditional probability, we write

$$P(\{T_n > u, Z_n = y\} \mid \{Z_{n-1} = x\} \cap H) = P(\{X_{S_n} = y\} \mid \{T_n > u, X_{S_{n-1}} = x\} \cap H) P(\{T_n > u\} \mid \{Z_{n-1} = x\} \cap H).$$

From the time homogeneity and strong Markov property applied to stopping time $S_{n-1}$, we get $P(\{T_n > u\} \mid \{Z_{n-1} = x\} \cap H) = \bar{F}_x(u)$. We further observe that $\{T_n > u, X_{S_{n-1}} = x\} \cap H = \{X_t = x, t \in S_{n-1} + [0, u]\} \cap H \in \mathcal{F}_{S_{n-1}+u}$. From the definition of excess time, we can write $S_n = S_{n-1} + u + Y_{S_{n-1}+u}$ for any $u \in [0, T_n]$. Again, applying the time homogeneity and Markov property of the continuous time Markov chain $X$, and the memoryless property of excess time $Y$, we obtain

$$P(\{X_{S_n} = y\} \mid \{T_n > u, X_{S_{n-1}} = x\} \cap H) = P(\left\{X_{S_{n-1}+u+Y_{S_{n-1}+u}} = y\right\} \mid \{X_{S_{n-1}+u} = x\}) = P_{xy}(0, Y_0).$$

This implies that sojourn time distribution and jump transision probabilities are independent. □

**Definition 1.17.** The jump process $Z$ is also sometimes referred to as the *embedded discrete time Markov chain* of the pure jump continuous time Markov chain $X$. The corresponding *jump transition probabilities* are defined for each state transition pair $x, y \in \mathcal{X}$ as

$$p_{xy} \triangleq P_{xy}(S_{n-1}, S_n) = P(\{X_{S_n} = y\} \mid \{X_{S_{n-1}} = x\}).$$

*Remark* 8. If $\nu_x = 0$, then for any $u \geqslant 0$, we have $P(\{Y_0 > u\} \mid \{X_0 = x\}) = 1$, and hence $S_1 = \infty$ almost surely whenever $X_0 = x$. By convention, we set $p_{xx} = 1$ and $p_{xy} = 0$ for all states $y \neq x$.

**Corollary 1.18.** *The transition probability matrix $p \triangleq (p_{xy} : x, y \in \mathcal{X})$ for embedded Markov chain $Z$ is stochastic, and if $\nu_x > 0$ then $p_{xx} = 0$.*

*Proof.* Recall $p_{xy} = P_{xy}(S_1)$. If $\nu_x > 0$, then $\lim_{u \to \infty} P(\{Y_0 > u\} \mid \{X_0 = x\}) = 0$, and hence $S_1$ is finite almost surely. By definition $X_{S_1} \neq X_0 = x$, and hence $p_{xx} = 0$. □

**Corollary 1.19.** *Consider a stable continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$. Then for all states $x, y \in \mathcal{X}$ and duration $u \in \mathbb{R}_+$,*

$$P(\{T_{n+1} > u, Z_{n+1} = y\} \mid \{X_0 = x_0, \dots, Z_n = x, S_0 \leqslant s_0, \dots, S_n \leqslant s_n\}) = p_{xy} e^{-u\nu_x}.$$

*Proof.* The history of the process until stopping time $S_n$ is given by $\mathcal{F}_{S_n} = \sigma(S_0, (Z_0, T_1), \dots, (Z_{n-1}, T_n), Z_n)$. Therefore $H \triangleq \{S_0 \leqslant s_0\} \cap_{i=1}^n \{Z_{i-1} = x_{i-1}, S_i \leqslant s_i\} \in \mathcal{F}_{S_n}$ and $\{Z_n = x\} \cap H \in \mathcal{F}_{S_n}$. Using strong Markov property and time-homogeneity of the continuous time Markov chain $X$, we have

$$P(\{T_{n+1} > u, Z_{n+1} = y\} \mid \{Z_n = x\} \cap H) = P_x \{S_1 > u, Z_1 = y\}.$$

The result follows from the previous Lemma 1.16. □

**Corollary 1.20.** *For a stable continuous time Markov chain, the jump transition probabilities* $(p_{xy} : x, y \in \mathcal{X})$ *and holding times* $T : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *are independent. The embedded jump process* $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ *is a homogeneous Markov chain with countable state space* $\mathcal{X}$. *The holding time sequence* $T : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *is independent and* $T_n$ *is distributed exponentially with rate* $\nu_{Z_{n-1}}$ *for each* $n \in \mathbb{N}$.

**Example 1.21 (Poisson process).** A Poisson counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with homogeneous rate $\lambda$ is a continuous time Markov chain with the countable state space is $\mathbb{Z}_+$ and uniform transition rate $\nu_n = \lambda$ for each state $n \in \mathbb{Z}_+$. It follows from the memoryless property of exponential random variables, that

$$\bar{F}_n(t) = P(\{Y_u > t\} \mid \{N_u = n\}) = P\{S_1 > t\} = e^{-\lambda t}.$$

Further, the embedded Markov chain or the jump process is given by the initial state $N_0 = 0$ and the transition probability matrix $P = (p_{n,m} : n, m \in \mathbb{Z}_+)$ where $p_{n,n+1} = 1$ and $p_{n,m} = 0$ for $m \neq n+1$. This follows from the definition of $T_1$, since $p_{n,m} = P(\{N_{T_1} = m\} \mid \{N_0 = n\}) = \mathbb{1}_{\{m=n+1\}}$.

# A Exponential random variables

**Lemma A.1.** *Let* $X$ *be an exponential random variable, and* $S$ *be any positive random variable, independent of* $X$. *Then, for all* $u \geqslant 0$

$$P(\{X > S + u\} \mid \{X > S\}) = P\{X > u\}.$$

*Proof.* Let the distribution of $S$ be $F$ and $X$ be memoryless with rate $\mu$. From the definition of conditional probability, we can write

$$P(\{X > S + u\} \mid \{X > S\}) = \frac{P\{X > S + u\}}{P\{X > S\}}.$$

Since a probability is an expectation of an indicator, we can write for all $u \geqslant 0$,

$$P\{X > S + u\} = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{X > S+u\}} \mid \sigma(S)]] = \mathbb{E}[e^{-\mu(S+u)}] = e^{-\mu u}\mathbb{E}[e^{-\mu S}].$$

It follows that $P\{X > S\} = \mathbb{E}e^{-\mu S}$ and since $P\{X > u\} = e^{-\mu u}$ for all $u \in \mathbb{R}_+$, the result follows. □

# B Coupling

For a regular and stable continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$, we denote the embedded Markov chain by $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ and the independent inter-jump time sequence by $T : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ where $T_n$ is an exponential random variable with rate $\nu_{Z_{n-1}}$ for all $n \in \mathbb{N}$. From the regularity and stability of process $X$, we have

$$0 < \overline{\nu} \leqslant \inf_{x \in \mathcal{X}} \nu_x \leqslant \sup_{x \in \mathcal{X}} \nu_x \leqslant \underline{\nu} < \infty.$$

Consider an *i.i.d.* uniform random sequence $U : \Omega \to [0,1]^{\mathbb{N}}$ and define three dependent random sequences $\underline{T}, T, \overline{T} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have

$$\overline{T}_n \triangleq -\frac{1}{\overline{\nu}} \log U_n, \qquad \underline{T}_n \triangleq -\frac{1}{\underline{\nu}} \log U_n, \qquad T_n \triangleq -\frac{1}{\nu_{Z_{n-1}}} \log U_n.$$

We observe that $\underline{T}$ and $\overline{T}$ are *i.i.d.* exponential random sequences with rates $\underline{\nu}$ and $\overline{\nu}$ respectively. Further, $T$ is an independent exponential random sequence with the rate $\nu_{Z_{n-1}}$ for $T_n$. In addition, by construction, we have $\underline{T}_n \leqslant T_n \leqslant \overline{T}_n$ for each $n \in \mathbb{N}$.