Lecture-29: Random Walks

1 Random walks

Definition 1.1. Let $\mathcal{X} \subseteq \mathbb{R}$, and consider a random step-size sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ such that $\mathbb{E}|X_n| < \infty$ for all $n \in \mathbb{N}$. The location of a particle after n steps is defined as $S_n \triangleq S_0 + \sum_{i=1}^n X_i$. The random sequence $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ is called a *random walk*. If $\mathcal{X} = \{-1, 1\}$, then the random walk is called **simple**.

Remark 1. Random walks are generalizations of renewal sequences. If step-size sequence X is a non-negative sequence indicating inter-event times, then S_n is the instant of the nth renewal event.

Definition 1.2. Consider a random sequence $Z : \Omega \to \mathbb{R}^{\mathbb{Z}_+}$ such that $Z_0 \triangleq 0$. We define $T_0 \triangleq 0$ and fix $k \in \mathbb{N}$, to inductively define the hitting time to kth low as

$$T_k \triangleq \inf\left\{n > T_{k-1} : Z_n \leqslant Z_{T_{k-1}}\right\} = T_{k-1} + \inf\left\{n \in \mathbb{N} : Z_{T_{k-1}+n} \leqslant Z_{T_{k-1}}\right\}. \tag{1}$$

The number of lows hit by process Z until time n is denoted by $N_n \triangleq \sum_{k \in \mathbb{N}} \mathbb{1}_{\{T_k \leqslant n\}}$, and $N : \Omega \to \mathbb{Z}_+^{\mathbb{Z}_+}$ is the counting sequence of number of lows of process Z.

Proposition 1.3. *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with an* i.i.d. *step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$. Let T_k be the time instant that random walk S hits kth low for each $k \in \mathbb{N}$, and N be the counting sequence for the number of lows of random walk S. If $\mathbb{E}X_1 > 0$, then $P\{T_1 = \infty\} > 0$ and $\mathbb{E}N_\infty < \infty$.*

Proof. Let $\mathcal{F}_\bullet$ denote the natural filtration of step-size sequence X such that $\mathcal{F}_n \triangleq \sigma(X_1, \dots, X_n)$ for each $n \in \mathbb{N}$. From the definition of the random time T_k, it is adapted to $\mathcal{F}_\bullet$ for each $k \in \mathbb{N}$.

Step 1. From the strong independence property for *i.i.d.* sequences, it follows that the distribution of $(X_{T_{k-1}+1}, \dots, X_{T_{k-1}+n})$ is identical to that of $(X_1, \dots, X_n)$ for any finite $n \in \mathbb{N}$, and independent of $\mathcal{F}_{T_{k-1}}$. Therefore, the distribution of $S_{T_{k-1}+n} - S_{T_{k-1}}$ is identical to that of S_n, and is independent of $\mathcal{F}_{T_{k-1}}$. Since we can write the difference $T_k - T_{k-1} = \inf\left\{n \in \mathbb{N} : \sum_{i=1}^n X_{T_k - 1 + i} \leqslant 0\right\}$, it follows that T_k is a random time, and $T_k - T_{k-1}$ is independent of $\mathcal{F}_{T_{k-1}}$ and distributed identically to T_1. Therefore, the sequence $(T_k - T_{k-1} : k \in \mathbb{N})$ is *i.i.d.* and $T : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ is a renewal sequence.

Step 2. We will show that $P\{T_1 < \infty\} < 1$ by showing that $\mathbb{R}_-$ is a set of transient states for Markov process S. From the L^1 strong law of large numbers, we have $\lim_{n \in \mathbb{N}} \frac{S_n}{n} = \mathbb{E}X_1 > 0$. Thus, $P\left\{\limsup_n S_n \leqslant 0\right\} = 0$ and the states in $\mathbb{R}_-$ are transient for random walk S.

Step 3. We observe that $\{T_k < \infty\} = \cap_{j=1}^k \{T_j < \infty\} = \cap_{j=1}^k \{T_j - T_{j-1} < \infty\}$. By the definition of counting sequence N and linearity of expectation, we get $\mathbb{E}N_\infty = \sum_{k \in \mathbb{N}} P\{T_k < \infty\}$. Since T is a renewal sequence, we get $\mathbb{E}N_\infty = \sum_{k \in \mathbb{N}} P\{T_1 < \infty\}^k = \frac{P\{T_1 < \infty\}}{P\{T_1 = \infty\}} < \infty$ and the result follows. $\square$

1.1 Duality in random walks

Lemma 1.4 (Duality principle). *Consider an exchangeable step-size sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$. The joint distributions of the finite sequence $(X_1, X_2, \cdots, X_n)$ and the reversed sequence $(X_n, X_{n-1}, \cdots, X_1)$ are identical for any finite $n \in \mathbb{N}$.*

Proof. Since $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ is an exchangeable sequence, the distribution of $(X_1, \dots, X_n)$ and $(X_{\alpha(1)}, \dots, X_{\alpha(n)})$ are identical for any n-permutation α. We apply this for a specific n-permutation $\alpha : [n] \to [n]$ such that $\alpha(i) \triangleq n - i + 1$ for each $i \in [n]$. The reversed sequence is $(X_{\alpha(1)}, \dots, X_{\alpha(n)})$ for this n-permutation. $\square$

Corollary 1.5. *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with exchangeable step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$.*
(a) The distributions of S_k and $S_n - S_{n-k}$ are identical for any $k \in [n]$.
(b) The joint distributions of random vectors $(S_1, \dots, S_n)$ and $(S_n - S_{n-1}, \dots, S_n)$ are identical for any $n \in \mathbb{N}$.

Proof. Since $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ is an exchangeable sequence, the distribution of $(X_1, \dots, X_n)$ and $(X_{\alpha(1)}, \dots, X_{\alpha(n)})$ are identical for any n-permutation α. We define an n-permutation $\alpha : [n] \to [n]$ such that $\alpha(i) \triangleq n - i + 1$ for each $i \in [n]$. We apply this for n-permutation $\alpha : [n] \to [n]$ such that $\alpha(i) \triangleq n - i + 1$ for each $i \in [n]$.

(a) We observe that $S_n - S_{n-k} = \sum_{i=1}^{k} X_{n-i+1} = \sum_{i=1}^{k} X_{\alpha(i)}$ for each $k \in [n]$. Since X is exchangeable and α is an n-permutation, the result follows.

(b) We observe that the sequence $(S_n - S_{n-1}, \dots, S_n) = (X_{\alpha(1)}, X_{\alpha(1)} + X_{\alpha(2)}, \dots, \sum_{i=1}^{n} X_{\alpha(i)})$, which is identically distributed to the sequence $(S_1, S_2, \dots, S_n)$ for exchangeable X and n-permutation α.

□

Proposition 1.6. *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with an* i.i.d. *step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$, and τ be the first hitting time of the random walk S to set of positive real numbers. If $\mathbb{E}X_1 > 0$ then $\mathbb{E}\tau < \infty$.*

Proof. We observe that $\tau \triangleq \inf\{n \in \mathbb{N} : S_n > 0\}$ is random time adapted to the natural filtration of step-size sequence X. From the definition of random time τ and duality principle, we can write

$$P\{\tau > n\} = P(\cap_{k=1}^{n} \{S_k \leqslant 0\}) = P(\cap_{k=1}^{n} \{S_n \leqslant S_{n-k}\}) = P\{S_n \leqslant \min\{0, S_1, \dots, S_{n-1}\}\}.$$

This implies that the process S hits a new low at time n, and in terms of the discrete process $T : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ defined in (1) for random walk S, we can write $\{\tau > n\} = \cup_{k \in \mathbb{N}} \{T_k = n\}$. Therefore,

$$\mathbb{E}\tau = 1 + \sum_{n \in \mathbb{N}} P\{\tau > n\} = 1 + \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} P\{T_k = n\} = 1 + \mathbb{E} \sum_{k \in \mathbb{N}} \mathbb{1}_{\{T_k < \infty\}} = 1 + \mathbb{E}N_\infty.$$

The result follows from the finiteness of $\mathbb{E}N_\infty$ from Proposition 1.3. □

1.2 Range of a random walk

Definition 1.7. Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with $S_0 \triangleq 0$. The number of distinct values of $(S_0, \cdots, S_n)$ is called **range**, denoted by $R_n \triangleq \left|\cup_{k=0}^{n} \{S_k\}\right|$. We define the first hitting time of random walk S to $x \in \mathbb{R}$ as the stopping time $\tau_x \triangleq \inf\{n \in \mathbb{N} : S_n = x\}$.

Proposition 1.8. *For a simple random walk, $\lim_{n \in \mathbb{N}} \frac{\mathbb{E}R_n}{n} = P\{\tau_0 = \infty\}$.*

Proof. We can define indicator function for S_k being distinct from $S_0, \dots, S_{k-1}$, as $I_k \triangleq \mathbb{1}_{\cap_{i=1}^{k} \{S_k \neq S_{k-i}\}}$. Then, we can write range R_n in terms of indicator I_k as $R_n = 1 + \sum_{k=1}^{n} I_k$. From the duality principle

$$P(\cap_{i=1}^{k} \{S_k \neq S_{k-i}\}) = P(\cap_{i=1}^{k} \{S_i \neq 0\}) = P\{\tau_0 > k\}, \quad k \in \mathbb{N}.$$

Therefore, $\mathbb{E}R_n = \sum_{k=0}^{n} P\{\tau_0 > k\}$, and the result follows from Cesàro mean. □

Theorem 1.9 (range). *For a simple random walk with $\mathbb{E}X_1 = 2p - 1$, $\lim_{n \in \mathbb{N}} \frac{\mathbb{E}R_n}{n} = 2(p \vee (1-p)) - 1$.*

Proof. From Proposition 1.8, it suffices to show that $P\{\tau_0 = \infty\} = |\mathbb{E}X_1|$.

Case 1. When $\mathbb{E}X_1 = 0$, we have $p = 1 - p = 1/2$, and simple random walk S is recurrent and hence $P\{\tau_0 = \infty\} = 0 = \mathbb{E}X_1$.

Case 2. When $\mathbb{E}X_1 > 0$, we have $p > 1 - p > 1/2$. From L^1 strong law of large numbers, we have $\lim_{n \in \mathbb{N}} \frac{S_n}{n} = \mathbb{E}X_1 > 0$ and hence $\{S_n \geqslant M \text{ infinitely often}\}$ almost surely for any $M \in \mathbb{R}_+$. It follows that 0 is a transient state and $P_x\{\tau_0 < \infty\} \in [0,1)$ for any $x > 0$. For each $x \in \mathbb{Z}$, we define $P_x\{\tau_0 < \infty\}$ to be the probability of simple random walk S hitting 0 in a finite time conditioned on $S_0 = x$. It follows that $P_{-1}\{\tau_0 < \infty\} = 1$ and we define $\beta \triangleq P_1\{\tau_0 < \infty\}$. From the law of total probability,

$$P\{\tau_0 < \infty\} = P\{\tau_0 < \infty, X_1 = 1\} + P\{\tau_0 < \infty, X_1 = -1\} = p\beta + (1-p).$$

To compute the conditional probability β, we apply law of total probability, definition of conditional probability, and Markov property of random walk S, to obtain

$$\beta = P_1\{\tau_0 < \infty\} = pP_2\{\tau_0 < \infty\} + (1-p)P_0\{\tau_0 < \infty\} = pP_2\{\tau_0 < \infty\} + (1-p).$$

From the Markov property and state transition homogeneity of random walk sequence, and definition of conditional probability, it follows that

$$P_2\{\tau_0 < \infty\} = P_2\{\tau_0 < \infty, \tau_1 < \infty\} = P_{S_{\tau_1}}\{\tau_0 < \infty\} P_2\{\tau_1 < \infty\} = (P_1\{\tau_0 < \infty\})^2 = \beta^2.$$

We conclude $\beta = \beta^2 p + 1 - p$, and since $\beta < 1$ due to transience, we get $\beta = \frac{1-p}{p}$, and hence the result follows.

Case 3. We can show similarly for the case when $\mathbb{E}X_1 < 0$.

□

2 Random walk for GI/GI/1 queues

Definition 2.1 ($GI/GI/1$ queue). Consider a single server queue with infinite buffer size and FCFS service discipline. We denote the random *i.i.d.* inter-arrival sequence by $\xi : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with an arbitrary common distribution $F : \mathbb{R}_+ \to [0,1]$. The random *i.i.d.* service time sequence is denoted by $\sigma : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with an arbitrary common distribution $G : \mathbb{R}_+ \to [0,1]$. We assume ξ and σ are independent and $\sigma_0 \triangleq 0$, to define *i.i.d.* step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ as $X_n \triangleq \sigma_{n-1} - \xi_n$ for each $n \in \mathbb{N}$. The waiting time before service for arrival n in the queue is denoted by W_n, where $W_0 \triangleq 0$ and for each $n \in \mathbb{N}$

$$W_n = (W_{n-1} + X_n) \vee 0. \tag{2}$$

For this $GI/GI/1$ queue, we associate a random walk sequence $S : \Omega \to \mathbb{R}^{\mathbb{Z}_+}$ with *i.i.d.* step-size sequence X, and we also define a random sequence $M : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ for all $n \in \mathbb{N}$, as

$$M_n \triangleq \max\{S_0, \dots, S_n\}.$$

Proposition 2.2. *Consider a GI/GI/1 queue with random waiting time sequence for arrivals denoted by $W : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and associated random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$. Then, we have for any $c \geqslant 0$*

$$P\{W_n \geqslant c\} = P\{M_n \geqslant c\} = P\left(\cup_{k\in[n]} \{S_k \geqslant c\}\right). \tag{3}$$

Proof. From the Lindley's recursion for waiting times and the definition of the associated random walk, we get $W_n = \max\{0, W_{n-1} + X_n\}$. Iterating the above relation with $W_0 = 0$, and using the definition of random walk S yields

$$W_n = \max\{0, X_n + \max\{0, W_{n-2} + X_{n-1}\}\} = \max\{0, X_n, X_n + X_{n-1} + W_{n-2}\} = \max\{0, S_n - S_{n-1}, \dots, S_n\}.$$

Using the duality principle for exchangeable random sequence X, we get $W_n = M_n$ in distribution. □

Corollary 2.3. *If $\mathbb{E}X_n \geqslant 0$, then we have $P\{W_\infty \geqslant c\} \triangleq \lim_{n\in\mathbb{N}} P\{W_n \geqslant c\} = 1$ for all $c \in \mathbb{R}$.*

Proof. It follows from Proposition 2.2 that $P\{W_n \geqslant c\}$ is non-decreasing in n and upper bounded by unity. Hence, by monotone convergence theorem, the limit exists and is denoted by $P\{W_\infty \geqslant c\} \triangleq \lim_{n\in\mathbb{N}} P\{W_n \geqslant c\}$. Therefore, by continuity of probability and Eq. (3), we have

$$P\{W_\infty \geqslant c\} = P\{S_n \geqslant c \text{ for some } n\}. \tag{4}$$

If $\mathbb{E}X_n = 0$, then the random walk is recurrent, and every state is almost surely reachable. If $\mathbb{E}X_n > 0$, then the random walk S will converge almost surely to positive infinity, from the L^1 strong law of large numbers. □

Remark 2. It follows from this corollary, that the stability condition $\mathbb{E}X_n < 0$ or $\mathbb{E}\sigma_{n-1} < \mathbb{E}\xi_n$ is necessary for the existence of a stationary distribution.

Proposition 2.4 (Spitzer's Identity). *Let $M_n \triangleq \max\{0, S_1, S_2, \dots, S_n\}$ for all $n \in \mathbb{N}$, then $\mathbb{E}M_n = \sum_{k=1}^n \frac{1}{k}\mathbb{E}S_k^+$.*

Proof. From the definition of M_n, we observe that

$$M_n \mathbb{1}_{\{S_n \leqslant 0\}} = M_{n-1}\mathbb{1}_{\{S_n \leqslant 0\}}, \qquad M_n \mathbb{1}_{\{S_n > 0\}} = \max\{S_1, S_2, \dots, S_n\}\mathbb{1}_{\{S_n > 0\}}.$$

Further, $\max\{S_1, S_2, \dots, S_n\} = X_1 + \max\{0, S_2 - S_1, \dots, S_n - S_1\}$. We define an n-permutation $\alpha : [n] \to [n]$ such that $\alpha(i) = i - 1$ for $i \in [n] \setminus \{1\}$ and $\alpha(1) = n$. Then from exchangeability of X, we have $(X_1 + \max\{0, S_2 - S_1, \dots, S_n - S_1\})\mathbb{1}_{\{S_n > 0\}}$ is equal in distribution to $(X_n + \max\{0, S_1, \dots, S_{n-1}\})\mathbb{1}_{\{S_n > 0\}} = (X_n + M_{n-1})\mathbb{1}_{\{S_n > 0\}}$. Combining all these results, we can write the mean of M_n as

$$\mathbb{E}M_n = \mathbb{E}M_n\mathbb{1}_{\{S_n \leqslant 0\}} + \mathbb{E}M_n\mathbb{1}_{\{S_n > 0\}} = \mathbb{E}M_{n-1}\mathbb{1}_{\{S_n \leqslant 0\}} + \mathbb{E}(X_n + M_{n-1})\mathbb{1}_{\{S_n > 0\}} = \mathbb{E}M_{n-1} + \mathbb{E}X_n\mathbb{1}_{\{S_n > 0\}}.$$

Since X is an *i.i.d.* sequence and $S_n = \sum_{i=1}^n X_i$, the tuple (X_i, S_n) has an identical joint distribution for all $i \in [n]$. Therefore, from the linearity of expectation and identical distribution of (X_i, S_n) for all $i \in [n]$, we get

$$\mathbb{E}S_n^+ = \mathbb{E}S_n\mathbb{1}_{\{S_n > 0\}} = \sum_{i=1}^n \mathbb{E}X_i\mathbb{1}_{\{S_n > 0\}} = n\mathbb{E}X_n\mathbb{1}_{\{S_n > 0\}} = n(\mathbb{E}M_n - \mathbb{E}M_{n-1}).$$

Since $M_n = M_1 + \sum_{i=1}^n (M_k - M_{k-1})$ and $M_1 = S_1^+$, the result follows from linearity of expectation and above result. □

Remark 3. Since $W_n = M_n$ in distribution, we have $\mathbb{E}[W_n] = \mathbb{E}[M_n] = \sum_{k=1}^n \frac{1}{k}\mathbb{E}[S_k^+]$.

3 Martingales for random walks

Proposition 3.1. *Consider an* i.i.d. *step-size sequence* $X : \Omega \to \mathbb{Z}^{\mathbb{N}}$ *such that* $|X_n| \leqslant M \in \mathbb{Z}_+$. *A random walk* $S : \Omega \to \mathbb{Z}^{\mathbb{N}}$ *with the step size sequence* X *is a recurrent Markov chain iff* $\mathbb{E}X_1 = 0$.

Proof. If $\mathbb{E}X_n \neq 0$, the random walk is clearly transient since it will diverge to $\pm\infty$ depending on the sign of $\mathbb{E}X_n$.

Conversely, if $\mathbb{E}X_n = 0$, then the random walk S is a martingale adapted to natural filtration $\mathcal{F}_\bullet$ of the step-size sequence. Assume that the random walk starts at state $S_0 = x \in \mathbb{Z}_+$. We define sets

$$A \triangleq \{-M, -M+1, \cdots, -2, -1\}, \qquad A_y \triangleq \{y+1, \ldots, y+M\}, \quad y > x.$$

Let $\tau \triangleq \inf\{n \in \mathbb{N} : S_n \in A \cup A_y\}$ denote the first hitting time by the random walk S to either A or A_y. It follows that τ is a random time with respect to $\mathcal{F}_\bullet$. Further, $\sup_{n\in\mathbb{N}} |S_{\tau\wedge n}| \leqslant y + M$. From the optional stopping theorem, we have $\mathbb{E}S_\tau = \mathbb{E}S_0 = x$. Thus, we have

$$x = \mathbb{E}_x S_\tau = \mathbb{E}_x[S_\tau \mathbb{1}_{\{S_\tau \in A\}} + S_\tau \mathbb{1}_{\{S_\tau \in A_y\}}] \geqslant -MP_x\{S_\tau \in A\} + y(1 - P_x\{S_\tau \in A\}).$$

Rearranging the above equation, we get a bound on probability of random walk S hitting A over A_y as

$$P_x\{S_n \in A \text{ for some } n\} \geqslant P_x\{S_\tau \in A\} \geqslant \frac{y-x}{y+M}.$$

Since the choice of $y \in \mathbb{Z}_+$ was arbitrary, taking limit $y \to \infty$, we see that for any $x \in \mathbb{Z}_+$, we have $P_x\{S_n \in A \text{ for some } n\} = 1$. Similarly for any $x \in \mathbb{Z}_-$, taking $B \triangleq \{1, 2, \cdots, M\}$ and $B_y \triangleq \{y - M, \ldots, y-1\}$ for $y < x$, we can show that $P_x\{S_n \in B \text{ for some } n\} = 1$. Result follows from combining the above two arguments to see that $P_x\{S_n \in A \cup B \text{ for some } n\} = 1$ for any $x \in \mathbb{Z}$. □

Proposition 3.2. *Consider a random walk* $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ *with* i.i.d. *step-size sequence* $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ *with common mean* $\mathbb{E}[X_1] \neq 0$. *For* $a, b > 0$, *we define the hitting time of the walk* S *exceeding a positive threshold* b *or going below a negative threshold* $-a$ *as*

$$\tau \triangleq \{n \in \mathbb{N} : S_n \geqslant b \text{ or } S_n \leqslant -a\}.$$

Let P_b *denote the probability that the walk hits a value greater than* b *before it hits a value less than* $-a$. *That is,* $P_b \triangleq P\{S_\tau \geqslant b\}$. *Then, for* $\theta \neq 0$ *such that* $\mathbb{E}e^{\theta X_1} = 1$, *we have* $P_b \approx \frac{1-e^{-\theta a}}{e^{\theta b} - e^{-\theta a}}$. *The above approximation is an equality when step size is unity and* a *and* b *are integer valued.*

Proof. For any $a, b > 0$, we can define stopping times

$$\tau_b = \inf\{n \in \mathbb{N} : S_n \geqslant b\}, \qquad \tau_{-a} = \inf\{n \in \mathbb{N} : S_n \leqslant -a\}.$$

Then, $\tau = \tau_b \wedge \tau_{-a}$, and we are interested in computing the probability $P_b = P\{\tau_b < \tau_{-a}\}$. We define a random sequence $Z : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ such that $Z_n \triangleq e^{\theta S_n}$ for all $n \in \mathbb{N}$, where $\theta \neq 0$ is chosen so that $\mathbb{E}e^{\theta X_1} = 1$. Hence, it follows that Z is a martingale with unit mean. We observe that $\sup_{n\in\mathbb{N}} |Z_{\tau\wedge n}| \leqslant e^{\theta b} \vee e^{-\theta a}$. From the optional stopping theorem, we get $\mathbb{E}e^{\theta S_\tau} = 1$. Thus, we get

$$1 = \mathbb{E}[e^{\theta S_\tau} \mathbb{1}_{\{\tau_b < \tau_{-a}\}}] + \mathbb{E}[e^{\theta S_\tau} \mathbb{1}_{\{\tau_b > \tau_{-a}\}}].$$

We can approximate $e^{\theta S_\tau} \mathbb{1}_{\{\tau_b < \tau_{-a}\}}$ by $e^{\theta b} \mathbb{1}_{\{\tau_b < \tau_{-a}\}}$ and $e^{\theta S_\tau} \mathbb{1}_{\{\tau_b > \tau_{-a}\}}$ by $e^{-\theta a} \mathbb{1}_{\{\tau_b > \tau_{-a}\}}$, by neglecting the overshoots past the thresholds b and $-a$. Therefore, we have $1 \approx e^{\theta b} P_b + e^{-\theta a}(1 - P_b)$. □

Corollary 3.3. *Let* $\tau \triangleq \tau_b \wedge \tau_{-a}$ *and* $P_b \triangleq P\{\tau_b < \tau_{-a}\}$, *then* $\mathbb{E}\tau \approx \frac{bP_b - a(1-P_b)}{\mathbb{E}X_1}$.

Proof. Repeat the above proof for zero mean martingale $(S_n - n\mathbb{E}X_1 : n \in \mathbb{N})$ to obtain $\mathbb{E}S_\tau = \mathbb{E}X_1 \mathbb{E}\tau$. Further, we approximate $S_\tau \mathbb{1}_{\{\tau_b < \tau_{-a}\}}$ by $b\mathbb{1}_{\{\tau_b < \tau_{-a}\}}$ and $S_\tau \mathbb{1}_{\{\tau_b > \tau_{-a}\}}$ by $-a\mathbb{1}_{\{\tau_b > \tau_{-a}\}}$ by neglecting the overshoots past the thresholds b and $-a$. □

A Strong independence property

Consider a filtration $\mathcal{F}_\bullet \triangleq (\mathcal{F}_t : t \in T)$ and a random time $\tau : \Omega \to T$ such that $\{\tau \leqslant t\} \in \mathcal{F}_t$ for all $t \in T$. The stopped α-algebra is defined as

$$\mathcal{F}_\tau \triangleq \{A \in \mathcal{F}_\infty : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t \text{ for all } t \in T\}.$$

Lemma A.1 (Strong independence property). *Let $X : \Omega \to \mathbb{R}^T$ be an* i.i.d. *process adapted to $\mathcal{F}_\bullet$ with distribution function $F : \mathbb{R}_+ \to [0,1]$. The process $Y : \Omega \to \mathbb{R}^T$ defined as $Y_t \triangleq X_{\tau+t}$ is independent of $\mathcal{F}_\tau$ and distributed identically to X.*

Proof. Consider $0 < t_1 < \cdots < t_m \in T$ and $(y_1, \ldots, y_m) \in \mathbb{R}^m$. Then, we observe that for any $A = \cap_{j=1}^n \left\{X_{s_j} \leqslant x_j\right\}$ for $s_n \leqslant \tau$, we have

$$P\Big(\cap_{k=1}^m \{Y_{t_k} \leqslant y_k\} \bigcap A \Big) = P\Big(\cap_{k=1}^m \{X_{\tau+t_k} \leqslant y_k\} \bigcap \cap_{j=1}^n \left\{X_{s_j} \leqslant x_j\right\} \Big) = \prod_{k=1}^m F(y_k) \prod_{j=1}^n F(x_j).$$

It follows that process Y is independent of $\mathcal{F}_\tau$ and distributed identically to X. □