1. Consider the following model for the flow of water in and out of a dam. Suppose during day n, Y_n units of water flow into the dam from outside sources such as rainfall and river flow. At the end of each day, water is released from the dam in the following manner.
   1. If the water content of the dam is greater than a, then the amount a is released.
   2. If it is less than or equal to a, then the total contents of the dam are released.

   The capacity of the dam is C. When the dam is full, any additional water that attempts to enter the dam is assumed lost. Thus, if the water level at the beginning of day n is x, then the level at the end of the day before any water is released is $(x + Y_n) \wedge C$. Let S_n denote the amount of water in the dam immediately after the water has been released at the end of day n. Assuming that the random sequence Y is *i.i.d.* , show that S is a random walk with reflecting barriers at 0 and $C - a$.

2. Consider the simple random walk S with *i.i.d.* step-size sequence $X : \Omega \to \{-1,1\}^{\mathbb{N}}$ having common mean $\mathbb{E}X_1 = 2p - 1 \leqslant 0$. For each $x \in \mathbb{Z}$, we define $\tau_x \triangleq \inf\{n \in \mathbb{N} : S_n = x\}$.
   (a) Show that τ_x is a random time adapted to natural filtration of X.
   (b) Show that $P\{\tau_k < \infty\} = \Big(P\{\tau_1 < \infty\}\Big)^k$.
   (c) Compute $P\{\tau_1 < \infty\}$.
   (d) Let θ^* be the unique positive root of equation $\mathbb{E}[e^{\theta X_1}] = 1$. Show that $e^{-\theta^*} = P\{\tau_1 < \infty\}$.

3. Consider a $GI/GI/1$ queue with *i.i.d.* inter-arrival time sequence $\xi : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, *i.i.d.* service time sequence $\sigma : \Omega \to \mathbb{R}_+^{\mathbb{Z}_+}$, and an initial arrival to an empty system at time 0. Define $X_i \triangleq \xi_i - \sigma_{i-1}$ for each $i \in \mathbb{N}$. For step-size sequence X, we define an associated forward random walk S, and backward random walk as $Z_i^n \triangleq S_n - S_{n-i}$ for each $i \in [n]$. Consider a sample path where $(x_1, \dots, x_6) = (1, -2, 2, -1, 3, -2)$.
   (a) Find the queueing delay for customer 6 as the maximum of the "backward" random walk with elements $0, Z_1^6, Z_2^6, \dots, Z_6^6$. Sketch this random walk.
   (b) Find the queueing delay for customers 1 to 5.
   (c) Which customers start a busy period (i.e., arrive when the queue and server are both empty)? Verify that if Z_i^6 maximizes the random walk in (a), then a busy period starts with arrival $6 - i$.
   (d) Sketch the forward walk for the sample path above and show that the queueing delay for each customer is the difference between two appropriately chosen values of this walk.

4. Consider a simple symmetric random walk S and the first upcrossing and downcrossing times for each threshold $x \in \mathbb{Z}$, as

   $$\tau_x^+ \triangleq \inf\{n \in \mathbb{N} : S_n \geqslant x\}, \qquad \tau_x^- \triangleq \inf\{n \in \mathbb{N} : S_n \leqslant x\}.$$

   Let $\tau \triangleq \tau_\alpha^+ \wedge \tau_\beta^-$ for integer thresholds $\beta < 0 < \alpha$.
   (a) Given that the random walk crosses β first, find an upper bound to the probability that α is crossed before another lower threshold at 2β is crossed.
   (b) Given that 2β is crossed before α, upper bound the probability that α is crossed before a threshold at 3β.
   (c) Extending this argument to successively lower thresholds, find an upper bound to each successive term, and hence obtain an upper bound on the overall probability that α is crossed.

(d) By observing that β is arbitrary, show that $P\{S_\tau \geqslant \alpha\} \leqslant \exp(-\theta^*\alpha)$ is valid with no lower threshold.

5. A real-valued random walk S with *i.i.d.* step-size sequence X having common probability density $f(x) \triangleq \frac{e^{-x}}{e-e^{-1}} \mathbb{1}_{\{|x| \leqslant 1\}}$.

   (a) Find the values of θ for which $g(\theta) \triangleq \mathbb{E}e^{\theta X_1} = 1$.

   (b) Let P_α be the probability that the random walk ever upcrosses a positive threshold $\alpha \in \mathbb{R}_+$. Find an upper bound to P_α of the form $P_\alpha \leqslant e^{-\alpha A}$, where A is a constant that does not depend on α. Find such a constant A.

   (c) Find a lower bound to P_α of the form $P_\alpha \geqslant Be^{-\alpha A}$, where A is the same as in (b) and B is a constant that does not depend on α.

6. Consider an *i.i.d.* random sequence $X : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ with common probability mass function $Q \in \mathcal{M}(\mathbb{Z})$. Assume that $Q_x > 0$ if and only if $|x| \leqslant 10$. Let S be a random walk associated with the step-size sequence X, and integer thresholds $\beta < 0 < \alpha$. Let τ_x^+, τ_x^- be the first upcrossing and downcrossing times, above and below threshold $x \in \mathbb{Z}$, respectively. We define $\tau \triangleq \tau_\alpha^+ \wedge \tau_\beta^-$, stopped random walk $(S_{n\wedge\tau} : n \in \mathbb{N})$, and $\pi^* \in \mathcal{M}(\mathbb{Z})$ defined as $\pi_k^* \triangleq P\{S_\tau = k\}$ for each $k \in \mathbb{Z}$.

   (a) Consider a Markov chain in which this stopped random walk is run repeatedly until the point of stopping. That is, the Markov chain transition probabilities are given by

$$P_{ij} = \begin{cases} Q_{j-i}, & \beta < i < \alpha, \\ 1, & i \notin \{\beta+1, \dots, \alpha-1\}, j = 0. \end{cases}$$

   All other transition probabilities are zero, and the set of states is the set of integers $[-9+\beta, 9+\alpha]$. Show that this Markov chain is ergodic.

   (b) Let $\pi \in \mathcal{M}(\mathbb{Z})$ be the steady-state probabilities for this Markov chain. Find the set of probabilities π^* for the stopping states of the stopped random walk in terms of π.

   (c) Find $\mathbb{E}S_\tau$ and $\mathbb{E}\tau$ in terms of π.