Real-Time Status Updates With Perfect Feedback Over Erasure Channels

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***Abstract*— Real-time decision making relies on the availability of accurate data and, therefore, delivering status updates in a timely fashion is of paramount importance. The topic of real-time status updates has received much attention in recent years. This article contributes new results to this research area by studying the interplay between average timeliness and design decisions made at the physical layer, for unreliable communication channels. Specifically, this study explores the tension between the fact that more reliable transmissions with lower probabilities of decoding failure tend to improve timely delivery, unless these improvements come at the expense of significantly longer codewords. The average timeliness is adopted as an evaluation criterion, and a framework to efficiently compute the performance of various transmission schemes for the binary erasure channel is developed. We show that the average timeliness decreases as we increase the feedback rate in a hybrid ARQ scheme for a range of codeword lengths. This article also provides design guidelines for the codeword length selection for an hybrid ARQ scheme to improve the average information timeliness. Numerical examples are included to further illustrate the applicability of our findings.**

***Index Terms*— Communication systems, low latency, status updates, block codes, forward error correction, feedback rate, hybrid ARQ.**

I. Introduction

THE wide availability of wireless sensors, microcontrollers, and actuators is changing the profile of typical wireless traffic. The traditional sustained connections attributable to human operators are being supplemented by a myriad of packet updates produced by machines, thereby creating heterogeneity in flows. The evolving character of wireless systems is an important component of the Internet of Things (IoT), a moniker often employed to describe next-generation networks. As sensing and actuation progressively expand to the wireless world, they are imposing new and distinct service requirements on existing communication infrastructures. For instance, cyber-physical systems depend on real-time status updates, relying on the latest telemetry data acquired by distributed devices for decision and control. Furthermore, mobile ad hoc networks need various kinds of status updates to know their neighborhood status, select routes, and schedule transmissions.

In recent years, researchers have introduced performance criteria to better understand the interplay between status update and communication systems. One of the guiding principles behind these new criteria is the fact that the timely delivery of information parcels is key in enabling smooth control and actuation. Stale information, on the other hand, can lead to incorrect decisions, greater residual errors, and instability. One specific performance criterion that has received much attention in the present context is the average age of information at the destination. This criterion captures the essence of staleness while admitting tractable problem formulations [2]–[5]. Owing to its popularity and ease of use, this is the performance criterion we adopt throughout. Information age is defined as the difference between current time t, and the time $U(t)$ at which the most recent status update was observed by the sensing device. Formally, we have the age process $A(t) \triangleq t - U(t)$. For a discrete-time setting, the limiting average of timeliness is defined as $\bar{A} \triangleq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} A(t)$.

In this article, we are focussing on timely status updates over unreliable channels, where the quality of communication links fluctuates over time. This is typical of several wireless settings, common communication medium in IoT, cyber-physical systems, and mobile ad-hoc networks. Traditionally, error correcting codes have been employed to protect sent data against channel impairments. With asymptotically long block lengths, it is possible to transmit data at rates that approach the Shannon capacity. Yet, such coding techniques entail undue delays and, therefore, may not be suitable for real-time status updates. Consequently, we explore the fundamental tension between data protection and delay in the context of real-time status updates, focusing on erasure channels. We are especially interested in the topic of remote sensing over wireless communication links. Both the areas of real-time status updates and coding for short block lengths have received attention

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over the past several years [1], [4], [6]–[14]. Concurrently, delay-sensitive communication has been investigated under error exponents, the normal approximation regime, and the moderate deviation regime [15], [16]. This high activity level points to the timeliness of the topic at hand.

Our aim is to combine results from these two areas by defining the communication channel at the symbol level and assessing the performance of the coded status update system using the average age of information criterion. This perspective is new; and it provides insight into the design of wireless links for status updates.

A. Background

Consider a scenario where a remote sensing device is monitoring a generic physical process X_t. We assume that every observation takes the form of a digital message containing exactly K bits of information. We consider a discrete time setting and assume that the sensor can observe the physical process at any point in time; and we call the corresponding sample a status update. The remote sensor must transmit the collected message to a central entity over an unreliable link akin to an unreliable channel. To protect the integrity of the measurements, it is natural to employ forward error correction. As is customary, suitable coding strategies will improve the probability of correctly recovering the sent message at the expense of additional redundancy bits in the transmitted codeword. Herein, we are especially interested in near real-time applications where the quality of a data sample is evaluated based on information staleness. This view point has become a common setting for real-time status updates.

The need to deliver messages in a timely manner prevents the use of long codewords. Rather, the problem formulation demands the application of coding strategies with low latency. Thus, a natural tension arises between the protection afforded by longer blocks, which translates into low probabilities of failure, and the ability of shorter codewords to deliver information with less latency when successfully decoded. The balance between these opposing considerations hinges, partly, on the character of the underlying channel. We additionally assume availability of reliable and instantaneous feedback from the receiver to the transmitter. This idealized assumption offers us the optimistic gains that can be achieved by feedback. In particular, we are interested in the impact of feedback on timeliness of received messages.

We restrict our attention primarily to coding schemes with finite block lengths. In particular, we explore limited feedback schemes such as hybrid automatic repeat request (hybrid ARQ) as a means of gracefully adapting to channel realizations. These schemes are known to perform well for data transmission over unreliable channels in the context of delay-sensitive applications. As such, they form an attractive option for the problem at hand as well. Figure 1 depicts the basic components of our envisioned system.

B. Related Work

Our problem formulation differs from previous contributions on real-time status updates in that it defines the operation of the channel at the symbol level. This enables us to explore the impact of physical layer design decisions on the average age. This framework enables the study of various coding schemes tailored to this application scenario. Our objective is to provide guidelines on system parameters for the aforementioned framework, and to compare the relative performance of different approaches. The specifics of our mathematical model are detailed in Section II.

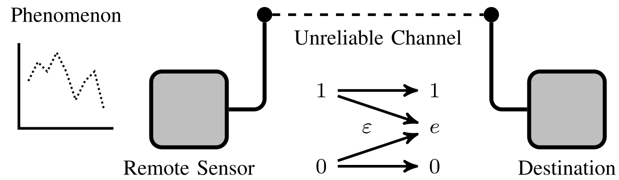


Fig. 1. This notional diagram offers an illustration of the system model, which is composed of a random phenomenon, a remote sensing device, a communication channel, and a data aggregator that receives status updates.

At this point, it is pertinent to note that there is abundant literature on the analysis of the age of information stemming from different communication models. The treatment of queuing theoretic models is considered in [5], [9], [12], [17]–[20]. In many cases, transmitting only the latest update can improve performance in terms of the age of information; accordingly, enhancements due to finite buffers and packet deadlines are presented in [21]–[24]. Various service profiles have also been investigated. For example, the authors study a generalized gamma service time distribution in [25]. In contrast to these contributions wherein the arrival of status updates is modeled by a stochastic process, our work adopts what is called a *generate at will* policy [26]–[28] under which the source can sample the latest status update of the observed phenomenon at any given time. In this latter setting, the performance of the age of information under ARQ and hybrid ARQ schemes, and its characterization from a channel coding perspective, are studied in [1], [29] and [28], [30]–[32]. Still, the literature on the age of information under the finite block length regime is not fully developed [33], [34], with opportunities for new insights.

C. Main Contributions

We consider a discrete information theoretic binary erasure channel for age-limited communication. We assume the source always has packets to send and, thus, system randomness originates from bit erasures. We consider hybrid ARQ for the transmission of updates over this unreliable channels, and characterize timeliness at the receiver. We summarize the main contributions of this article below.

We analytically show that, in certain regimes, there exists a natural tradeoff between the average feedback rate and average timeliness at the receiver. In particular, we show that for a constrained set of hybrid ARQ codeword lengths, if the codewords are refined then the average timeliness is improved at the receiver, at the cost of increased feedback rate.

Finding optimal hybrid ARQ codes is a computationally challenging problem, because the objective function of average age is non-convex and the optimizing variables in the form of codeword lengths are constrained to take on integer values. Nevertheless, we identify a class of hybrid ARQ codes that capture this tradeoff, that are shown to be near-optimal in empirical studies.

We emphasize that the proposed model first appeared in [1], and is now a customary model for age analysis as evidenced by subsequent works [33], [35]–[37]. Najm *et al.* present the optimal age for erasure channel without feedback in [37]. Age with feedback is considered in [1], where the results indicate that the age performance of ARQ is worse than that of a fixed length coded update with no-retransmission whenever ARQ employs the re-transmission of the same codeword. Contrastingly, in this work, we find that hybrid ARQ can significantly outperform the fixed length scheme when the sizes of incremental redundancy sub-blocks are chosen judiciously. Thus, we have shown that hybrid ARQ can outperform ARQ and fixed-length coded update with no retransmission.

Our findings shed new light on hybrid ARQ as it pertains to the age of information. The ensuing guidelines for system design constitute a significant departure from previous work.

D. Organization

We introduce the system model in Section II, that describes in detail the channel model in Section II-A, the hybrid ARQ scheme in Section II-B, and performance metrics and the problem statement in Section II-C. Renewal process associated with the proposed hybrid ARQ scheme is introduced in Section III, which aids in computing the corresponding average age and average feedback rate in Section III-B. We demonstrate in Section III-C that hybrid ARQ codes have smaller average age than a fixed length coded update with no retransmission, and reformulate the problem statement as an integer optimization problem in Section III-D. We present our main structural results in Section IV, where we show the impact of hybrid ARQ refinement on average age in Section IV-A and average feedback rate in Section IV-B. Numerical results are provided in Section V, and the article is concluded in Section VI.

II. System Model

The phenomenon being monitored is modeled as a sequence of independent and uniformly distributed symbols. The sensing device is observing a process $M(t) \in \{0, \dots, 2^K - 1\}$ for all $t \in \mathbb{Z}_+$. The size of an observation is K information bits, irrespective of the past. After it is acquired, the observation must be communicated to a central location using a wireless link. In this paper, we do not consider source coding strategies such as joint source-channel coding or data compression based on differential encoding. The design and evaluation of such advanced schemes are typically tied to specific applications. The use of a generic source instead enables this work to focus on the tradeoff we wish to explore. It also offers a suitable mathematical framework that renders analysis tractable. We use the notation $\mathbb{Z}_+$ for non-negative integers, the notation $\mathbb{N}$ for positive integers, and the notation $[m]$ to represent the set $\{1, 2, \dots, m\}$ for any positive integer $m \in \mathbb{N}$.

A. Channel Model

We adopt a channel model commonly found in the information theory literature, namely the bit-wise memoryless

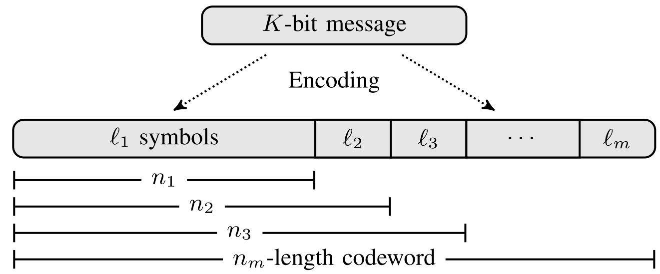


Fig. 2. In a hybrid ARQ scheme, a K-bit source message is encoded into a N-length codeword. The codeword is further divided into sub-blocks of varied lengths, which are sequentially transmitted up to an ACK message from the receiver.

binary erasure channel. Additionally, we assume that each bit transmission over this channel requires one unit of time. The choice of this channel is motivated by its analytical tractability for average age analysis, and is a first step in the direction of more complex physical layer channel models. The channel is driven by an independent and identically distributed (*i.i.d.*) Bernoulli process $(\zeta_t \in \{0, 1\} : t \in \mathbb{N})$ with mean $\mathbb{E}\zeta_t = \varepsilon$. In terms of the process sample ζ_t at time t, we can write the channel output $Y_t \in \{0, 1, e\}$ for binary channel input $X_{t-1} \in \{0, 1\}$ as $Y_t = X_{t-1}(1 - \zeta_t) + e\zeta_t$, where symbol e denotes an erased bit. Hence, every transmitted bit is received at the destination with probability $1 - \varepsilon$ and it is erased with probability ε, independently of other bits.

Remark 1: The number of erasures in received bits in time-slots $\{t + 1, \dots, t + n\}$ is given by $\sum_{i=1}^{n} \zeta_{t+i}$, and it has a binomial distribution with parameters (n, ε).

B. Hybrid ARQ

We denote the transmission time of kth source message by t_k, and denote the kth message by $M_k \triangleq M(t_k)$. We consider an incremental redundancy scheme using an (N, K) forward error correcting block code denoted by the map $c : \{0, 1\}^K \to \{0, 1\}^N$. We attempt to transmit this encoded message $c(M_k)$ using at-most m rounds, where we assume an immediate and error-free single-bit feedback from the receiver to the source at the end of each round. Bit 0 indicates a decoding failure or the negative acknowledgment (NACK) and bit 1 indicates a decoding success or the positive acknowledgment (ACK). Accordingly, the N-length codeword $c(M_k)$ is divided into m sub-blocks, each of length ℓ_i for potential transmission in round $i \in [m]$. We use n_i to denote the number of encoded bits transmitted by the end of round i, i.e., $n_i \triangleq \sum_{j=1}^{i} \ell_j$ for all $i \in [m]$. This yields $\ell_1 = n_1 < \cdots < n_m \leqslant N$. The encoding structure is depicted in Fig. 2.

We employ $\xi_{k,i}$ to indicate decoding success for transmitted message k in round i or earlier. Equivalently, $\xi_{k,i}$ is equal to one when an ACK regarding message k is received by round i, and it is zero otherwise. Given receiver feedback from round i, the source takes one of two possible actions.

1) If $\xi_{k,i} = 0$ and $i < m$, then the source starts round $i + 1 \in [m]$ and transmits an additional sub-block of length ℓ_{i+1} corresponding to the kth codeword $c(M_k)$.
2) If either $\xi_{k,i} = 1$ or $i = m$, then the transmission of message k halts. The source collects new observation

$M_{k+1} = M(t_{k+1})$. It then encodes this observation and, subsequently, initiates the transmission of the first sub-block of length ℓ_1 of the corresponding codeword $c(M_{k+1})$.

Remark 2: We recall that at the end of round i for message k, the receiver has received n_i-length channel output corresponding to first n_i bits of the codeword $c(M_k)$. We can consider the trailing $N - n_i$ bits of the kth codeword to be erased, and denote the effective set of erasures until round i for message k, by $E_{k,i} \triangleq \{j \in [n_i] : \zeta_{t_k+j} = 1\} \cup \{n_i + 1, \ldots, N\}$. Since $\zeta_t \leqslant 1$ for all $t \in \mathbb{N}$, we have $E_{k,1} \supseteq \cdots \supseteq E_{k,m}$. The received N-length codeword at time $t + n_i$ is $y^i \in \{0, 1, e\}^N$ where

$$y^i_j = c(M_k)\mathbb{1}_{\{j \notin E_{k,i}\}} + e\mathbb{1}_{\{j \in E_{k,i}\}}.$$

Remark 3: For the erasure channel, the decoder $d : \{0, 1, e\}^N \to \{0, 1\}^K \cup \{f\}$ maps the N-length channel output to the K-bit transmitted message or declares a failure f. For message k, the indicator of decoding success after round i is $\xi_{k,i}$. Recall that the number of erasures as a function of round i is non-increasing in i. Then, under optimal decoding, the indicator of decoding success $\xi_{k,i}$ is non-decreasing with round $i \in [m]$.

Remark 4: When incremental redundancy is utilized in conjunction with the binary erasure channel introduced in Section II-A, an important quantity is the probability $F(n_i) = \mathbb{E}\xi_{k,i}$ that the received sequence is decodable in round i or earlier. Since $\xi_{k,i}$ is non-decreasing with round i, it follows that $F(n_i)$ is a non-decreasing function of i. We also note that $\bar{F}(n_m) = P\{\xi_{k,m} = 0\}$ is the probability that the kth transmitted message is not successfully decoded. That is, the probability of getting a NACK in round i is $\bar{F}(n_i) \triangleq 1 - F(n_i)$.

C. Performance Metric

Our performance criterion is timeliness or, equivalently, staleness. This is defined as the difference between current time t and the time $U(t)$ at which the most recent status update was observed by the sensing device. Formally, we have the age process

$$A(t) \triangleq t - U(t). \quad (1)$$

We can define the limiting average information age as

$$\bar{A} \triangleq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} A(t).$$

We also measure the limiting average feedback rate from the receiver to the transmitter. Let $N_F(t)$ denote the number of feedback messages until time t, then the limiting average is given by $\bar{Z} \triangleq \limsup_{T\to\infty} \frac{1}{T} N_F(T)$. Figure 3 illustrates a sample path of the age process for a hybrid ARQ incremental redundancy scheme where 3-bit messages are encoded into 5-bit codewords. Every message transmission attempt over this channel takes place within at most 3 rounds, with $(n_1, n_2, n_3) = (3, 4, 5)$.

Our objective is to design the block sizes $(\ell_1, \ldots, \ell_m)$ such that we can minimize the limiting average age subject to keeping the average feedback rate below a certain threshold ρ. In practice, we will choose an underlying (N, K)-code to provide incremental redundancy. Since the probability of success is 0 for any $n_i < K$ and the maximum codeword length is N, we can assume that $K \leqslant n_1 < n_2 < \cdots < n_m \leqslant N$. That is, we restrict our attention to the following set of block assignment vectors

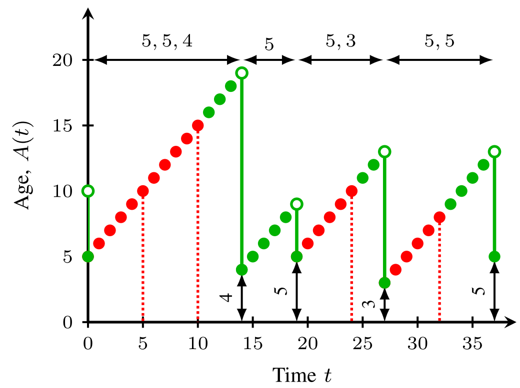


Fig. 3. This figure depicts a sample path of the age process for the incremental redundancy scheme with parameters $K = 3, N = 5, m = 3$ and $(n_1, n_2, n_3) = (3, 4, 5)$. The parts of the age trajectory marked in red and green indicates a codeword failure and success, respectively. The first update is received at $t = 5$ but fails to get decoded. Similarly, the second update leads to a decoding failure. Consequently, the age increases linearly until a decoding success occurs with the third update. Notice that the third status update is successfully decoded with 4 bits. The instantaneous age decreases to 4, at time $t = 14$, subsequently the fourth update is received and so forth.

$$B_0 \triangleq \{(n_1, \ldots, n_m) : K \leqslant n_1 < \cdots < n_m \leqslant N\}. \quad (2)$$

Using B_0 and feedback rate threshold ρ, we can formally state the optimization problem of interest.

Problem 1: Find a finite block assignment vector $\mathbf{n} \in B_0$ for the following optimization problem,

$$\begin{aligned} &\underset{\mathbf{n}\in B_0}{\text{minimize}}\ \bar{A}(\mathbf{n}) \\ &\text{subject to } \bar{Z}(\mathbf{n}) \leqslant \rho. \end{aligned}$$

Our design goal is to select $\mathbf{n}$ (or, equivalently, block sizes $(\ell_1, \ldots, \ell_m)$) as to minimize average age, while maintaining the average number of feedback messages below a prescribed threshold $\rho \in [0, 1]$. We note that a threshold set to $\rho = 1$ essentially means unconstrained feedback. As target ρ is lowered, the maximum admissible feedback rate decreases. In the next section, we proceed with the derivation of expressions for the average age $\bar{A}$ and the average feedback rate $\bar{Z}$ as function of assignment vector $\mathbf{n}$.

III. Renewal Process and Limiting Averages

Let $N_0 = 0$. We can define the number of codeword receptions until the kth decoding success as $N_k \triangleq \inf\{j > N_{k-1} : \xi_{j,m} = 1\}$. Then, we can write the number of codeword failures between two successful decoding as $R_k \triangleq N_k - N_{k-1} - 1$. Let V_k denote the round in which codeword N_k gets decoded, i.e., $V_k \triangleq \inf\{i \in [m] : \xi_{N_k,i} = 1\}$.

Lemma 2: The random sequences $(R_k : k \in \mathbb{N})$ and $(V_k : k \in \mathbb{N})$ are independent, and they are both i.i.d. *with respective* *distributions*

$$P\{R_k = r\} = F(n_m)\bar{F}(n_m)^r, \quad r \geqslant 0,$$
$$P\{V_k = i\} = \frac{F(n_i) - F(n_{i-1})}{F(n_m)}, \quad i \in [m].$$

Proof: See Appendix A. ■

Corollary 3: The first and the second moments for random variable R_k are

$$\mathbb{E}R_k = \frac{\bar{F}(n_m)}{F(n_m)}, \ \mathbb{E}R_k^2 = \frac{\bar{F}(n_m)^2 + \bar{F}(n_m)}{F(n_m)^2}.$$

The pth moment of random variable n_{V_k} for $p \geqslant 1$ is

$$\mathbb{E}n_{V_k}^p = n_m^p - \sum_{i=1}^{m-1}(n_{i+1}^p - n_i^p)\frac{F(n_i)}{F(n_m)}.$$

Remark 5: Based on the properties of the Geometric distribution, one can verify that $\mathbb{E}R_k^2 - 2(\mathbb{E}R_k)^2 = \mathbb{E}R_k$.

A. Renewal Process

With $S_0 = 0$, we can recursively define the time-instant of the kth successful reception as

$$S_k = S_{k-1} + n_m R_k + n_{V_k}. \tag{3}$$

The time-interval between the $(k-1)$th and the kth successful decoding event is denoted by

$$T_k \triangleq S_k - S_{k-1} = n_m R_k + n_{V_k}. \tag{4}$$

Since the random *i.i.d.* sequences $(R_k : k \in \mathbb{N})$ and $(V_k : k \in \mathbb{N})$ are independent and have finite first and second moments, it follows that the sequence $(T_k : k \in \mathbb{N})$ is also *i.i.d.* with finite first and second moments and, hence, $(S_k : k \in \mathbb{N})$ is a renewal sequence. We note that the renewals occur at the instants of successful decoding of a codeword. We present the first two moments of the inter-renewal times T_k in the following lemma.

Lemma 4: The first and the second moments of the inter-renewal time T_k are given by

$$\mathbb{E}T_k = \frac{n_m}{F(n_m)} - \sum_{i=1}^{m-1}(n_{i+1} - n_i)\frac{F(n_i)}{F(n_m)},$$
$$\mathbb{E}T_k^2 = -\sum_{i=1}^{m-1}\frac{F(n_i)}{F(n_m)}(n_{i+1} - n_i) \left[n_{i+1} + n_i + 2n_m\frac{\bar{F}(n_m)}{F(n_m)}\right] + \frac{n_m^2(1 + \bar{F}(n_m))}{F(n_m)^2}.$$

Proof: This result follows from the independence of the *i.i.d.* sequences $(R_k : k \in \mathbb{N})$ and $(V_k : k \in \mathbb{N})$, and their first and second moments presented in Corollary 3. ■

B. Average Age and Average Feedback

The generation time $U(t)$ of the latest successfully decoded codeword only changes upon decoding success, $(S_k : k \in \mathbb{N})$. Furthermore, the kth successfully received codeword was generated at time $U(S_k) = S_k - n_{V_k}$. Thus, for any time t in the kth renewal interval $I_k \triangleq \{S_{k-1}, \dots, S_k - 1\}$ we have

$$U(t) = U(S_{k-1}) = S_{k-1} - n_{V_{k-1}} \quad t \in I_k. \tag{5}$$

Using (5) for the generation time, we write the age at the receiver as a function of time t as

$$A(t) = t - U(S_{k-1}) = t - S_{k-1} + n_{V_{k-1}} \quad t \in I_k. \tag{6}$$

Lemma 5: For the incremental redundancy described in Section II-B, the limiting empirical average age is almost surely

$$\bar{A} = \frac{\mathbb{E}\sum_{t \in I_k} A(t)}{\mathbb{E}T_k} = \frac{\mathbb{E}T_k^2}{2\mathbb{E}T_k} + \mathbb{E}n_{V_k} - \frac{1}{2}. \tag{7}$$

Proof: See Appendix B. ■

Let $N_F(t)$ denote the number of feedback messages until time t. Recall that the receiver sends a one-bit feedback message per sub-block of the hybrid ARQ codeword. Hence, the number of feedback messages in kth renewal interval is $N_F(S_k) - N_F(S_{k-1}) = mR_k + V_k$.

Lemma 6: The limiting average number of feedback messages is $\bar{Z} = \frac{m\mathbb{E}R_k + \mathbb{E}V_k}{\mathbb{E}T_k}$ almost surely.

Proof: We can write the limiting average of number of feedback messages as

$$\lim_{T \to \infty} \frac{N_F(T)}{T} = \lim_{T \to \infty} \left(\frac{N(T)}{T}\right) \frac{\sum_{k=1}^{N(T)}(N_F(S_k) - N_F(S_{k-1}))}{N(T)}.$$

The result follows from an application of strong law of large numbers. ■

C. Comparison of Hybrid ARQ With Fixed Length Scheme

The fixed length scheme can be regarded as a special case of the hybrid ARQ scheme, where the number of rounds $m = 1$ and we denote the codeword length for the fixed-length scheme by $n_1 = n_m = N$. We represent the limiting empirical average age for a fixed N-length scheme by $\bar{A}_f(N)$.

Corollary 7: The average age for fixed length scheme with codeword length N is almost surely

$$\bar{A}_f(N) = \frac{N-1}{2} + \frac{N}{F(N)}. \tag{8}$$

Proof: For the fixed length status update $V_k = 1 = m$, and hence the block-length at the time of success is $n_{V_k} = N$. Therefore, the mean and second moment of the inter-renewal times reduce to

$$\mathbb{E}T_k = \frac{N}{F(N)}, \quad \mathbb{E}T_k^2 = \frac{N^2(1 + \bar{F}(N))}{F(N)^2}.$$

The desired result is obtained by substituting these two expressions in the limiting average age of (7) in Lemma 5. ■

We first compare the limiting average age performance of a fixed N-length codeword with an incremental redundancy scheme where $n_m = N$. We show that for any given codeword with N bits employed for both the schemes, the limiting average age under any hybrid ARQ scheme is lower than that of the fixed-length scheme.

Lemma 8: Let $\bar{A}_f(N)$ and $\bar{A}(\mathbf{n})$ be the limiting average age of the fixed length scheme with codeword length N and of the hybrid ARQ scheme with block assignment vector $\mathbf{n} \in B_0$, respectively. If the hybrid ARQ scheme has $m = |\mathbf{n}|$ rounds with total codeword length $n_m = N$, then $\bar{A}_f(N) \geqslant \bar{A}(\mathbf{n})$ with equality if and only if $\mathbf{n} = \{N\}$.

Proof: See Appendix C. ■

D. Integer Optimization Problem

Given that we have obtained expressions for the limiting average age and limiting average of feedback rate, we can rewrite Problem 1 explicitly in terms of the *i.i.d.* renewal period length $T_k = n_m R_k + n_{V_k}$, the number of codeword failures R_k, and the round of success V_k.

Problem 9: Find a finite block assignment vector $\mathbf{n} \in B_0$ for the following optimization problem,

$$\begin{aligned} &\underset{\mathbf{n}\in B_0}{\text{minimize}}\ \bar{A}(\mathbf{n}) = \mathbb{E}n_{V_k} + \frac{\mathbb{E}T_k^2}{2\mathbb{E}T_k} - \frac{1}{2} \\ &\text{subject to } \bar{Z}(\mathbf{n}) = \frac{m\mathbb{E}R_k + \mathbb{E}V_k}{\mathbb{E}T_k} \leqslant \rho. \end{aligned}$$

The objective and the constraint are both functions of vector $\mathbf{n}$ in Problem 1. This optimization problem is an integer program with non-convex objective function, and the optimizing variable $\mathbf{n}$ (the codeword block-lengths and rounds) is constrained to take finitely many values. We note that, in general, there are no efficient algorithm to solve generic integer programming problems over the set of all finite integer sequences.

IV. HYBRID ARQ REFINEMENT

In this section, we derive structural results on the limiting average of age $\bar{A}$ and limiting feedback $\bar{Z}$ as functions of arbitrary block assignment vector $\mathbf{n}$. Based on these general guidelines, we solve Problem 1 for a constrained set of block assignment vectors $\mathbf{n}$. We numerically show that this restricted class of block assignment vectors $\mathbf{n}$ is near-optimal, in the sense that the minimum limiting average age under this class is close to the one found by searching among all possible vectors $\mathbf{n}$ that ensure the limiting average feedback rate is below threshold ρ.

To this end, we investigate the impact of refinement of a block assignment vector $\mathbf{n}'$ on two performance metrics, average age and average feedback rate. A block assignment vector $\mathbf{n}$ is called the *refinement* of $\mathbf{n}'$, if $\mathbf{n}' \subseteq \mathbf{n}$ with lengths $|\mathbf{n}'| \leqslant |\mathbf{n}|$. Intuitively, it may seem that a finer block assignment vector $\mathbf{n}$ would lead to a lower average age since one can stop opportunistically, and higher average feedback rate since one sends a larger number of ACK/NACK messages to achieve this. However, it turns out that our intuition regarding the average age is not entirely correct, as is illustrated by Example 12, where $\bar{A}(\mathbf{n}') \leqslant \bar{A}(\mathbf{n})$ for a refinement $\mathbf{n}$ of $\mathbf{n}'$. Thus, the intuition that *refinement necessarily reduces age* does not hold for all integer sequences $\mathbf{n}'$ and their refinements. Before presenting Example 12, we introduce some notation which we use throughout the remainder of the article in Notation 10, and a specific forward error correction policy for hybrid ARQ implementation in Example 11.

Notation 10: Recall that for block assignment vector $\mathbf{n}$, we denote the number of codeword failures before the kth successful reception by R_k, the number of rounds for the kth successful reception by V_k, and the time-period between the $(k-1)$th and the kth successful receptions by T_k. The corresponding notations for the block assignment vector $\mathbf{n}'$ are R'_k, V'_k, T'_k. We will adopt this notation throughout the paper, whenever we compare the performance of two block assignment vectors $\mathbf{n}$ and $\mathbf{n}'$. The codeword length for two block assignment vectors are denoted by $m = |\mathbf{n}|$ and $m' = |\mathbf{n}'|$, respectively.

Example 11 (hybrid ARQ using random linear codes): We assume that the forward error correction is implemented using a *permutation invariant* and *monotone* code. That is, the conditional probability of decoding failure given a set of erasures E (denoted by $P_f(N, K, E)$) depends only on the number of erasures and not their locations, and it increases with number of erasures. For an (N, K) permutation invariant code, the conditional probability of decoding failure given ℓ erasures in the received codeword is denoted by $P_f(N, K, \ell)$. In particular, for an (N, K) random linear code [38], [39], we have

$$P_f(N, K, \ell) = \left(1 - \prod_{i=0}^{\ell-1}\left(1 - 2^{i-N+K}\right)\right)\mathbb{1}_{\{\ell \leqslant N-K\}}.$$

For incremental redundancy achieved using such codes with block-length $\mathbf{n}$, the probability of decoding success in round i is given by $F(n_i) = \mathbb{E}P_f(N, K, L_i + N - n_i)$, where $L_i = \sum_{j=1}^{n_i}\zeta_{t+j}$ is the random number of erased bits in the n_i-length codeword and the $N - n_i$ trailing codeword bits are effectively assumed to be erased. The expectation is taken over random erasures, where the channel erasure indicators $(\zeta_t \in \{0,1\} : t \in \mathbb{N})$ are assumed to be *i.i.d.* Bernoulli with $\mathbb{E}\zeta_t = \varepsilon$ and, consequently, L_i has a binomial distribution with parameter (n_i, ε).

Example 12 (A case where refinement does not reduce age): Consider a hybrid ARQ scheme employing an $(N, K) = (200, 10)$ random linear code over a binary erasure channel with erasure probability $\varepsilon = 0.1$. We consider two block assignment vectors $\mathbf{n} = (n_1, n_2, n_3) = (10, 12, 200)$ and $\mathbf{n}' = (n_2, n_3) = (12, 200)$, such that $\mathbf{n}$ is a refinement of $\mathbf{n}'$. We show that the mean age for the refined block vector $\mathbf{n}$ is larger than the mean age for the block vector $\mathbf{n}'$.

From the decoding success distribution function $F(\cdot)$ for the block assignment vector $\mathbf{n}$, we define the following variables

$$\alpha_1 \triangleq F(n_1), \quad \alpha_2 \triangleq F(n_2) - F(n_1), \quad \alpha_3 \triangleq F(n_3) - F(n_2).$$

Using Example 11, we can compute $(F(n_i) : i \in [3])$ for the given block vector $\mathbf{n} = (10, 12, 100)$ and $\varepsilon = 0.1$, and hence we obtain the probabilities $(\alpha_1, \alpha_2, \alpha_3) = (0.35, 0.3, 0.35)$.[1]

[1]Note that $F(n_3) = \sum_{i=1}^{3}\alpha_i$ is only approximately 1 since there is a positive probability of decoding failure at codeword length $n_3 = 200$ albeit negligibly small.

In terms of the values $(\alpha_1, \alpha_2, \alpha_3)$, we can write the decoding success probability for hybrid ARQ transmissions with both the block assignment vectors $\mathbf{n}$ and $\mathbf{n}'$ as $F(n_3) = \sum_{i=1}^{3} \alpha_i$. The mean number of failures in a renewal interval for both the block assignment vectors remain same, since the probability of decoding success $F(n_3)$ for a single hybrid ARQ transmission is identical for both options. Specifically, we have

$$\mathbb{E}R_k = \mathbb{E}R'_k = \frac{\bar{F}(n_3)}{F(n_3)}, \quad \mathbb{E}R_k^2 = \mathbb{E}(R'_k)^2 = \frac{\bar{F}(n_3)^2 + \bar{F}(n_3)}{F(n_3)^2}.$$

The probability mass function for the number of rounds until success for block assignment vector $\mathbf{n}$ is given by $P_{V_k} = \left(\frac{\alpha_1}{\sum_{i=1}^3 \alpha_i}, \frac{\alpha_2}{\sum_{i=1}^3 \alpha_i}, \frac{\alpha_3}{\sum_{i=1}^3 \alpha_i}\right)$. The corresponding probability mass function for the number of rounds until success for block assignment vector $\mathbf{n}'$ is $P_{V'_k} = \left(\frac{\alpha_1+\alpha_2}{\sum_{i=1}^3 \alpha_i}, \frac{\alpha_3}{\sum_{i=1}^3 \alpha_i}\right)$. The first two moments of number of bits transmitted in a successful hybrid ARQ transmission, for the block assignment vector $\mathbf{n}$, are given by

$$\mathbb{E}n_{V_k} = \frac{\sum_{i=1}^3 \alpha_i n_i}{\sum_{i=1}^3 \alpha_i}, \quad \text{and } \mathbb{E}n_{V_k}^2 = \frac{\sum_{i=1}^3 \alpha_i n_i^2}{\sum_{i=1}^3 \alpha_i}.$$

Likewise, the values for the block assignment vector $\mathbf{n}'$ are given by

$$\mathbb{E}n'_{V'_k} = \frac{(\alpha_1+\alpha_2)n_2 + \alpha_3 n_3}{\sum_{i=1}^3 \alpha_i}, \text{and}$$
$$\mathbb{E}(n'_{V'_k})^2 = \frac{(\alpha_1+\alpha_2)n_2^2 + \alpha_3 n_3^2}{\sum_{i=1}^3 \alpha_i}.$$

From the definition of T_k in (4), the fact that first two moments of R_k and R'_k are equal for two block assignment vectors $\mathbf{n}$ and $\mathbf{n}'$, and defining $d_1 \triangleq \mathbb{E}n'_{V'_k} - \mathbb{E}n_{V_k}$, we can write

$$\mathbb{E}T'_k = d_1 + \mathbb{E}T_k, \quad \text{and } \mathbb{E}(T'_k)^2 = (n_2 + n_1)d_1 + \mathbb{E}T_k.$$

For our choice of system parameters, we observe that $d_1 = \frac{\alpha_1(n_2-n_1)}{\sum_{i=1}^2 \alpha_i}$. Using the fact that $\mathbb{E}R_k^2 = 2(\mathbb{E}R_k)^2 + \mathbb{E}R_k$ and from the computation of the limiting empirical average age in Lemma 5, we can compute the difference $\bar{A}(\mathbf{n}') - \bar{A}(\mathbf{n})$ between the limiting empirical average of age for the two block assignment vectors $\mathbf{n}$ and $\mathbf{n}'$, as

$$\frac{\mathbb{E}T_k(\mathbb{E}T_k + d_1 + n_1 + n_2 - n_m) + \mathbb{E}(n_{V_k}(n_m - n_{V_k}))}{2\mathbb{E}T_k(\mathbb{E}T_k + d_1)/d_1}.$$

We see that the denominator is always positive, and it is possible to make the numerator negative if n_m is very large and n_1 and n_2 are roughly equal, and much smaller than n_m. In this case, we have $F(n_m) \approx 1$ and hence $\mathbb{E}R_k \approx 0$ is very small with $\sum_{i=1}^m \alpha_i \approx 1$. Taking $n_2 = n_1 + 1$, we can write the age difference as

$$\bar{A}(\mathbf{n}') - \bar{A}(\mathbf{n}) \approx \frac{\mathbb{E}n_{V_k}(\alpha_1 + 2n_1 + 1) - \mathtt{Var}\left[n_{V_k}\right]}{2(\mathbb{E}n_{V_k} + d_1)\mathbb{E}n_{V_k}/d_1}.$$

Thus, in this setting of large n_m and small n_1, n_2, if we additionally have $\frac{\mathtt{Var}[n_{V_k}]}{\mathbb{E}n_{V_k}} \geqslant F(n_1) + 2n_1 + 1$, then it follows that $A(\mathbf{n}') \leqslant A(\mathbf{n})$, and the refinement does not reduce age. For our choice of system parameters, we numerically computed the difference between limiting empirical average age, and found that $\bar{A}(\mathbf{n}') - \bar{A}(\mathbf{n}) = -0.18 \leqslant 0$.

A. Impact of Refinement on Average Age

We have shown that indeed it is not *always* true that sending a refined block assignment vector would decrease the age. This was shown keeping fixed n_m, the total number of bits sent for complete transmission of block assignment vector $\mathbf{n}$. However, it turns out that *refinement can reduce age under certain sufficient conditions*, and we next present such sufficient conditions.

Theorem 13: Consider two block assignment vectors $\mathbf{n}' \subseteq \mathbf{n}$ with lengths $m' \leqslant m$, respectively, such that $n_m = n'_{m'}$ and $n_1 \geqslant \frac{n_m}{4}$. Then, the limiting empirical average age for the two block assignment vectors satisfy $\bar{A}(\mathbf{n}') \geqslant \bar{A}(\mathbf{n})$.

Proof: See Appendix D. ■

In the proof of Theorem 13, we see that for any refinement $\mathbf{n}' \supseteq \mathbf{n}$ of the block vector $\mathbf{n}$ such that $n_m = n'_{m'}$, the number of rounds until success for the refined vector is stochastically dominated with $V'_k \leqslant V_k$. Therefore, for each $k \in \mathbb{N}$, we have $\mathbb{E}n'_{V'_k} \leqslant \mathbb{E}n_{V_k}$ and $\mathbb{E}(n'_{V'_k})^2 \leqslant \mathbb{E}n_{V_k}$. Furthermore, we observe that the numbers of decoding failures in a renewal interval remain identical in distribution for both the refined vector $\mathbf{n}'$ and the original vector $\mathbf{n}$ because $n_m = n'_{m'}$. Consequently, for each $k \in \mathbb{N}$, we have $\mathbb{E}R_k = ER'_k$ and $\mathbb{E}R_k = \mathbb{E}(R'_k)^2$. We also stress that the length of the kth renewal interval $T_k = n_m R_k + n_{V_k}$, where n_{V_k} and R_k are independent random variables. It follows that $\mathbb{E}T'_k \leqslant \mathbb{E}T_k$ and $\mathbb{E}(T'_k)^2 \leqslant \mathbb{E}T_k^2$. From Lemma 5, we know that the limiting empirical average of age for a block vector $\mathbf{n}$ is given by

$$\bar{A}(\mathbf{n}) = \mathbb{E}n_{V_k} + \frac{\mathbb{E}T_k^2}{2\mathbb{E}T_k} - \frac{1}{2}.$$

From this expression, it follows that the first term $\mathbb{E}n_{V_k}$ decreases with refinement. However, both the numerator and the denominator in the second term decrease with refinement. Thus, it is not immediately clear whether the second term increases or decreases with refinement. The condition $4n_1 \geqslant n_m$ in Theorem 13 suffices to guarantee that a refinement improves average age.

B. Impact of Refinement on Feedback Rate

The previous section casts average age minimization as a constrained optimization problem. Due to the nonlinearity of the objective function and the discrete nature of the feasible and constraint sets, the optimal integer solution to Problem 1 remains elusive. Nevertheless, we found that a refinement of the block assignment vector between its start and end points always improves average age when $n_1 \geqslant n_m/4$. Intuitively, it seems that refining blocks may increase the average feedback rate, and thence may lead to violation of feedback rate constraint; this possibility warrants a closer look. In this section, we examine the impact of refining a block assignment on average feedback rate. In Lemma 14, we show that the average feedback rate can only become larger when the block assignment vector is refined, while keeping the total codeword length fixed.

Lemma 14: Consider two block assignment vectors $\mathbf{n}' \subseteq \mathbf{n}$ with lengths m' and m, respectively, such that $n_m = n'_{m'}$. The limiting average feedback rate for the two block assignment vectors satisfy $\bar{Z}(\mathbf{n}') \leqslant \bar{Z}(\mathbf{n})$.

Proof: See Appendix E. ■

In words, Lemma 14 asserts that, given a fixed codeword length, subdividing hybrid ARQ blocks increases the average feedback rate. With n_1 and n_m fixed, the most refined block assignment vector is $\mathbf{n} = (n_1, n_1+1, \ldots, n_m)$, where $n_m = n_1 + m - 1$. As a related result, Lemma 15 states that the average feedback rate keeps increasing for such block assignment vectors with m, for a fixed n_1.

Lemma 15: The average feedback rate is monotonically increasing in codeword length for all sequences $\mathbf{n} \subseteq \mathbb{N}$ such that $\mathbf{n} = n_1 - 1 + [m]$, with given $n_1 \in \mathbb{N}$.

Proof: See Appendix F. ■

V. Optimal Block Assignment Vector

We return to the constrained optimization introduced in Problem 1. This consists of finding the block assignment vector $\mathbf{n} \subset \mathbb{N}$ that minimizes average age for a hybrid ARQ system over an *i.i.d.* erasure channel, subject to the constraint that the average feedback rate remains below threshold ρ. As mentioned before, since the objective and the constraint are both functions of integer-valued block assignment vector $\mathbf{n}$, this optimization problem can be viewed as an integer program. In general, it is computationally challenging to find the optimal solution to such problems.

However, we were able to derive certain structural properties for the given integer constrained optimization problem in the previous two sections. We define the following set of ordered block assignment vectors

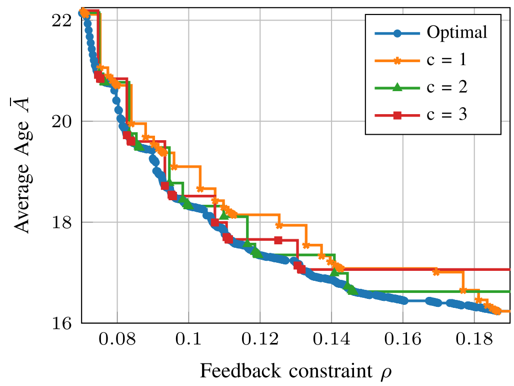
$$B_1 \triangleq \left\{\mathbf{n} \in B_0 : n_1 \geqslant \frac{n_m}{4}, n_m = N\right\}.$$

In Section IV, we concluded that the limiting average age decreases when we refine any block-assignment vector $\mathbf{n} \in B_1$. We further concluded in Section IV-B that the average feedback rate increases when we refine the block-assignment vector $\mathbf{n} \in B_0 \supset B_1$. This implies that, if the feedback rate constraint ρ is large enough, then the optimal block assignment vector within B_1 is of the form $\mathbf{n} \in B_2$, where

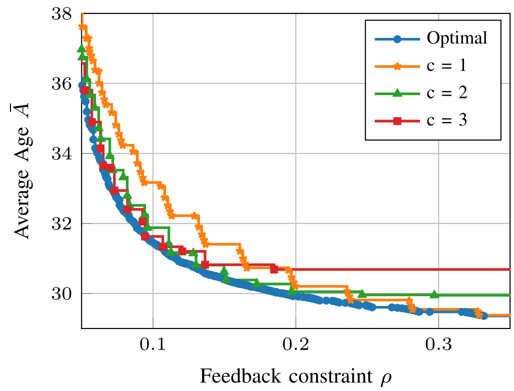
$$B_2 \triangleq \{\mathbf{n} \in B_1 : \mathbf{n} = n_1 - 1 + [m], m = N + 1 - n_1\}.$$

To find the optimal block assignment vector in the absence of feedback rate constraint, we must identify the optimal starting point $n_1 \in \mathbb{N}$.

Again, for a block assignment vector $\mathbf{n} = (n_1, \ldots, n_m) \in B_0$, we offer structural results for refinements $\mathbf{n}' \supseteq \mathbf{n}$ where $n_m = n'_{m'}$ and $4n_1 \geqslant n_m$. Ideally, we would like to understand the impact that an extension of the form $\mathbf{n}'' = (n_1, \ldots, n_m, n_{m+1}) \in B_0$ may have on the limiting average age in the absence of feedback constraints. Unfortunately, this is a complicated question because extensions do not preserve the distribution of R_k. Rather, the answer seems to depend intimately on the code structure and the channel parameters. This explains, partly, our focus on refinements rather than extensions. That is, it remains unclear whether the optimal block assignment vector that is a solution to Problem 1 has any specific structure. To gain a better understanding, we numerically study a system where the source has $K = 10$ bits of information to send in every time slot. The selected system employs an hybrid ARQ scheme with random linear codes for block encoding. We consider the channel to be *i.i.d.* bit-wise binary symmetric erasure, and consider two different erasure probabilities $\epsilon \in \{0.05, 0.4\}$. We find the optimal solution to the constrained integer optimization problem defined in Problem 1 by searching over all possible block assignment vectors $\mathbf{n} \in B_0$, where the set B_0 is defined in (2). This solution has the lowest limiting empirical average age for a fixed limiting empirical average feedback rate. We choose the starting point for the block assignment vector as $n_1 \geqslant K$ because a receiver can never decode a K-bit message with fewer than K binary symbols. Further, the maximum length of the random linear code is chosen to be $N = 30$, due to computational considerations. These optimal block assignments are used as benchmarks for our study of structured solutions.



(a) Low erasure probability, $\epsilon = 0.05$.



(b) High erasure probability, $\epsilon = 0.4$.

Fig. 4. Performance of periodic hybrid ARQ schemes compared to optimal allocations when using random linear codes with $N = 30$, $K = 10$.

Owing to the vast number of possible refinements to a fixed-length codewords, we confine our attention to a restricted class of block assignment vectors $B_3(c)$ where $c \in \mathbb{N}$ and

$$B_3(c) \triangleq \left\{\mathbf{n} \in B_0 : n_i = n_1 + (i-1)c, i \in [m], n_1 \geqslant \frac{n_m}{4}\right\}.$$

That is, the block assignment vectors depend on the starting point n_1, the periodicity of increase c, and the number of steps m. We notice that $B_3(c)$ is a generalization of class B_2, since the set $B_3(c)$ reduces to B_2 when the periodicity $c = 1$ and $n_m = N$.

Figure 4 plots the limiting average age with respect to the limiting average feedback rate for the optimal block assignment vectors found by exhaustive search over the set B_0, for erasure probabilities $\epsilon \in \{0.05, 0.4\}$ in Fig. 4(a) and Fig. 4(b), respectively. We also show the performance of the best block assignment vector within the class of periodic block assignment vectors. Specifically, the graphs include average age versus average feedback rate curves for periodic block assignment vectors $\mathbf{n} \in B_3(c)$ where $c \in \{1, 2, 3\}$.

For the low erasure probability case depicted in Fig. 4(a), we observe that the initial block is very long and the message is decoded successfully with high probability when the average feedback constraint is very stringent. In this regime, sub-partitioning beyond the first block is not crucial and. consequently, several schemes offer comparable performance. For the high erasure probability case depicted in Fig. 4(b), we gather that a larger period should be adopted when the average feedback is very stringent. As the feedback constraint becomes looser, while keeping the same erasure probability, it becomes advantageous to switch to smaller periods.

VI. Conclusion and Future Work

In this article, we have characterized the limiting average age for hybrid ARQ schemes employed over point-to-point binary symmetric erasure channels. We showed that judiciously chosen incremental redundancy schemes do better than the fixed length codes when there is feedback from the receiver to the transmitter in terms of ACK/NACK messages. We next demonstrated that under certain conditions on the block assignment vector, refinement reduces the limiting empirical average age. However, this comes at the cost of increased feedback from the receiver. In particular, block refinement increases the limiting average feedback rate.

A natural question for a system designer is to find the block assignment vector that minimizes the average age given a constraint on the average feedback rate. However, this discrete optimization problem is an integer program and finding the optimal solution remains computationally challenging. Still, numerical results suggest that it is sufficient to restrict our attention to the class of periodic block assignment vectors, as they offer a performance level close to the optimal solution. Altogether, our work provides pragmatic guidelines for choosing incremental redundancy schemes for timely communication.

There are several avenues of future research in this area. We have adopted a symmetric binary erasure channel model; the definition of age should be extended to channels with errors. Further, we have characterized timeliness performance for a simple *i.i.d.* channel model. In fact, we anticipate the gains in average age to be higher for channels with memory. The quest for additional algorithmic structures and the characterization of timeliness gains for correlated channels are other interesting potential research directions for future work.

Appendix A
Proof of Lemma 2

We can write the event consisting of r decoding failures before the kth decoding success as $\{R_k = r\} = \{\xi_{N_k,m} = 1\} \cap_{j=1}^{r} \{\xi_{(N_{k-1}+j),m} = 0\}$. This expression follows from the *i.i.d.* structure of the erasure channel and the fact that the probability of a codeword failure is $1 - F(n_m)$. Accordingly, we can write the probability of the event that the kth successfully decoded word was decoded in round i by

$$P\{V_k = i\} = \frac{P(\{\xi_{N_k,m} = 1, \xi_{N_k,i} = 1\} \cap_{j=1}^{i-1} \{\xi_{N_k,j} = 0\})}{P\{\xi_{N_k,m} = 1\}}.$$

From the monotonicity of indicators $(\xi_{k,i} : i \in [m])$, it follows that $\{\xi_{N_k,i} = 1\} \subseteq \{\xi_{N_k,m} = 1\}$ for $i \leqslant m$ and $\{\xi_{N_k,i-1} = 0\} \subseteq \{\xi_{N_k,j} = 0\}$ for all $j \leqslant i-1$. Further, we can write the set $\{\xi_{N_k,i} = 1\}$ as a disjoint union $\{\xi_{N_k,i} = 1\} = \{\xi_{N_k,i} = 1, \xi_{N_k,i-1} = 0\} \cup \{\xi_{N_k,i-1} = 1\}$. Summarizing the above results, we get

$$\begin{aligned} &\{\xi_{N_k,m} = 1, \xi_{N_k,i} = 1\} \cap_{j=1}^{i-1} \{\xi_{N_k,j} = 0\} \\ &= \{\xi_{N_k,i-1} = 0, \xi_{N_k,i} = 1\} = \{\xi_{N_k,i} = 1\} \setminus \{\xi_{N_k,i-1} = 1\}, \end{aligned}$$

where the event $\{\xi_{N_k,i-1} = 1\} \subseteq \{\xi_{N_k,i} = 1\}$. The result follows from evaluating the probability of the events on both sides.

Appendix B
Proof of Lemma 5

We can write the cumulative sum of age in the kth renewal interval I_k as $C_k \triangleq \sum_{t \in I_k} A(t)$. Using the expression for age $A(t)$ at time t in (6), along with the definition for the kth renewal interval I_k, we obtain the cumulative sum of age

$$C_k = \sum_{t=S_{k-1}}^{S_k - 1} (t - S_{k-1} + n_{V_k}) = \frac{T_k(T_k - 1)}{2} + n_{V_{k-1}} T_k. \tag{9}$$

Since V_{k-1} is independent of V_k and R_k, it is also independent of T_k. Therefore, we can write the mean cumulative sum of age in the kth renewal interval I_k as $\mathbb{E}C_k = \frac{1}{2}\mathbb{E}T_k^2 + \mathbb{E}T_k \left(\mathbb{E}n_{V_{k-1}} - \frac{1}{2}\right)$. Let $N(T)$ be the number of renewals until time T. Then, we have the following upper and lower bounds for the empirical average age $\frac{1}{T}\sum_{k=1}^{N(T)} C_k \leqslant \frac{1}{T}\sum_{t=0}^{T} A(t) \leqslant \frac{1}{T}\sum_{k=1}^{N(T)+1} C_k$. To decouple the summands, we can partition the sum $\sum_{k=1}^{N(T)} C_k$ into odd and even renewals, such that

$$\sum_{k=1}^{N(T)} C_k = \sum_{k=1}^{\lfloor N(T)/2 \rfloor} C_{2k} + \sum_{k=1}^{\lfloor N(T)/2 \rfloor + 1} C_{2k-1}.$$

Dividing both sides by the aggregate time T and taking the limit $T \to \infty$, we get

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{N(T)} C_k \\ &= \lim_{T\to\infty} \frac{N(T)}{2T} \Big(\lim_{T\to\infty} \frac{2}{N(T)} \sum_{k=1}^{\lfloor N(T)/2 \rfloor} C_{2k} \\ &\quad + \lim_{T\to\infty} \frac{2}{N(T)} \sum_{k=1}^{\lfloor N(T)/2 \rfloor + 1} C_{2k-1} \Big). \end{aligned}$$

From the strong law of large numbers, we gather that $\lim_{T\to\infty} N(T)/T = 1/\mathbb{E}T_k$ almost surely. Further, since $(C_{2k} : k \in \mathbb{N})$ and $(C_{2k-1} : k \in \mathbb{N})$ are *i.i.d.* sequences,[2] applying the strong law of large number, we get the following almost sure equality

$$\lim_{T\to\infty} \frac{1}{T}\sum_{k=1}^{N(T)} C_k = \lim_{k\to\infty} \frac{1}{2\mathbb{E}T_k}\left(\mathbb{E}C_{2k} + \mathbb{E}C_{2k-1}\right).$$

We can similarly partition the sum $\frac{1}{N(T)+1}\sum_{k=1}^{N(T)+1} C_k$ to analyze the limiting behavior of the upper bound on the empirical average age. The result follows from the fact that $\mathbb{E}C_{2k} = \mathbb{E}C_{2k-1}$ for $k \geqslant 2$.

Appendix C
Proof of Lemma 8

Let $|\mathbf{n}| = m$, then the block assignment vector $\mathbf{n} = (n_1, \dots, n_m)$ is an ordered sequence of m positive integers, with total codeword length $n_m = N$. Denoting $n_0 = 0$, we recall that the number of bits sent in the ith round is $\ell_i = n_i - n_{i-1}$ for round $i \in [m]$. For $i \in [m-1]$, we denote the scaled number of bits sent in the $(i+1)$th round as $d_i \triangleq \ell_{i+1}\frac{F(n_i)}{F(N)} \geqslant 0$.

We can rewrite $\mathbb{E}n_{V_k}$ and $\mathbb{E}T_k$ in terms of the scaled number of bits $(d_i : i \in [m-1])$ and the total codeword length N as $\mathbb{E}\mathbf{n}_{V_k} = N - \sum_{i=1}^{m-1} d_i$, and $\mathbb{E}T_k = \frac{N}{F(N)} - \sum_{i=1}^{m-1} d_i$. The second moment of inter-renewal time becomes

$$\mathbb{E}T_k^2 = \frac{N^2(1+\bar{F}(N))}{F(N)^2} - \sum_{i=1}^{m-1} d_i\left(n_{i+1} + n_i + 2N\frac{\bar{F}(N)}{F(N)}\right).$$

We obtain the limiting empirical average age for the hybrid ARQ scheme with the block assignment vector $\mathbf{n}$ by substituting the above expressions for $\mathbb{E}n_{V_k}$, $\mathbb{E}T_k$, and $\mathbb{E}T_k^2$ in Lemma 5. Further, we can get the limiting empirical age for fixed length codeword N from Corollary 7. Therefore, we can write the difference as

$$\bar{A}_f(N) - \bar{A}(\mathbf{n}) = \sum_{i=1}^{m-1} d_i + \frac{\sum_{i=1}^{m-1} d_i \frac{(n_{i+1}+n_i-N)}{2}}{\frac{N}{F(N)} - \sum_{i=1}^{m-1} d_i}.$$

Denoting $F(n_0) = 0$, we recall that n_{V_k} is a random variable with probability mass function $P_{V_k}(i) = P\{n_{V_k} = n_i\} = \frac{F(n_i)-F(n_{i-1})}{F(N)}$ for each $i \in [m]$. We can verify that

$$\sum_{i=1}^{m-1} d_i = N - \mathbb{E}n_{V_k}, \quad \sum_{i=1}^{m-1} d_i(n_{i+1}+n_i) = N^2 - \mathbb{E}n_{V_k}^2.$$

Therefore, we can write $\sum_{i=1}^{m-1} d_i(n_{i+1} + n_i - N) = \mathbb{E}[n_{V_k}(N - n_{V_k})] \geqslant 0$. Hence, it follows that the difference $\bar{A}_f(N) - \bar{A}(\mathbf{n}) \geqslant 0$ from the positivity of scaled differences d_i and the positivity of denominator $\mathbb{E}T_k$ for the second term.

[2] Note that C_{2k} and C_{2k-1} are dependent, however we only need the individual sequences to be *i.i.d.* and not the two sequences to be independent.

Appendix D
Proof of Theorem 13

For the block assignment vectors $\mathbf{n}$ and $\mathbf{n}'$ defined in the Theorem 13, we use the notation R_k, V_k, T_k and R_k', V_k', T_k' respectively, as defined in Notation 10.

Step 1. $R_k = R_k'$ in distribution. We note that $F(n_m) = F(n'_{m'})$ because $n_m = n'_{m'}$. Therefore, the distribution of R_k and R_k' are identical. From Corollary 3, it follows that the first two moments of R_k and R_k' are identical for every $k \in \mathbb{N}$.

Step 2. Reduction to single refinement. Let $\mathbf{n} = (n_1, \dots, n_m)$ and $t \in [m-1]$, then it suffices to show that for $\mathbf{n}' = (n_1, \dots, n_{t-1}, n_{t+1}, \dots, n_m)$, we have $\bar{A}(\mathbf{n}') \geqslant \bar{A}(\mathbf{n})$.

Step 3. Relation between n_{V_k} and $n'_{V_k'}$. We denote the probability mass function of V_k by $P_{V_k} = \left(\frac{F(n_i)-F(n_{i-1})}{F(n_m)} : i \in [m]\right)$. Since $\mathbf{n}$ is a one-level refinement of $\mathbf{n}'$, we can write

$$n_i' = n_i \mathbb{1}_{\{i \leqslant t-1\}} + n_{i+1}\mathbb{1}_{\{i \geqslant t\}}, \quad i \in [m-1]. \tag{10}$$

We can express the probability mass function of $V_k' \in [m-1]$ in terms of P_{V_k}, for $i \in [m-1]$, as

$$P_{V_k'}(i) = \begin{cases} P_{V_k}(i), & i \leqslant t-1, \\ P_{V_k}(t+1) + P_{V_k}(t), & i = t, \\ P_{V_k}(i+1), & i \geqslant t+1. \end{cases} \tag{11}$$

From the definition of moments, the form of the probability mass functions, and the order on $\mathbf{n}$; we get the differences

$$d_1 \triangleq \mathbb{E}[n'_{V_k'} - n_{V_k}] = (n_{t+1} - n_t)P_{V_k}(t) \geqslant 0 \tag{12}$$

$$d_2 \triangleq \mathbb{E}[(n'_{V_k'})^2 - n_{V_k}^2] = (n_{t+1}^2 - n_t^2)P_{V_k}(t) \geqslant 0 \tag{13}$$

for this specific choice of block assignment vectors $\mathbf{n}$, $\mathbf{n}'$.

Step 4. Relation between the two limiting empirical average ages $\bar{A}(\mathbf{n})$ and $\bar{A}(\mathbf{n}')$. From Lemma 5, we can write the difference between the empirical averages for two block assignment vectors $\mathbf{n}'$ and $\mathbf{n}$ as

$$\bar{A}(\mathbf{n}') - \bar{A}(\mathbf{n}) = d_1 + \frac{\mathbb{E}T_k^2 + 2n_m\mathbb{E}R_k d_1 + d_2}{2(\mathbb{E}T_k + d_1)} - \frac{\mathbb{E}T_k^2}{2\mathbb{E}T_k}.$$

Since $d_2 = (n_{t+1} + n_t)d_1$, we can write the scaled difference

$$\begin{aligned}&(\bar{A}(\mathbf{n}') - \bar{A}(\mathbf{n}))\frac{\mathbb{E}T_k}{d_1}(\mathbb{E}T_k + d_1) \\ &= \mathbb{E}T_k(2\mathbb{E}T_k + 2d_1 + 2n_m\mathbb{E}R_k + n_{t+1} + n_t) - \mathbb{E}T_k^2.\end{aligned}$$

From Remark 5, we have $\mathbb{E}R_k^2 - 2(\mathbb{E}R_k)^2 = \mathbb{E}R_k$ and, therefore, $\mathbb{E}T_k^2 = 2n_m\mathbb{E}R_k\mathbb{E}T_k + n_m(\mathbb{E}T_k - \mathbb{E}n_{V_k}) + \mathbb{E}n_{V_k}^2$. It follows that the scaled difference

$$\begin{aligned}&(\bar{A}(\mathbf{n}') - \bar{A}(\mathbf{n}))\frac{\mathbb{E}T_k}{d_1}(\mathbb{E}T_k + d_1) = \mathbb{E}T_k(2\mathbb{E}T_k + 2d_1 \\ &+ n_{t+1} + n_t - n_m) + \mathbb{E}n_{V_k}(n_m - n_{V_k}).\end{aligned}$$

Furthermore, we have $d_1 \geqslant 0$, $n_m \geqslant n_{V_k} \geqslant n_1$, $\mathbb{E}T_k \geqslant \mathbb{E}n_{V_k} \geqslant n_1$, and $n_t \geqslant n_1$ for all $t \in [m-1]$. Consequently, it follows that

$$(\bar{A}(\mathbf{n}') - \bar{A}(\mathbf{n}))\frac{\mathbb{E}T_k}{d_1}(\mathbb{E}T_k + d_1) \geqslant \mathbb{E}T_k(2n_1 + 2n_1 - n_m).$$

That is, under the hypothesis $4n_1 \geqslant n_m$, we have the desired relation between the two limiting empirical average ages.

Appendix E
Proof of Lemma 14

For the block assignment vectors $\mathbf{n}$ and $\mathbf{n}'$ defined in Lemma 14, we use the notation R_k, V_k, T_k, m and R'_k, V'_k, T'_k, m' respectively, as defined in Notation 10. Since the block assignment vectors $\mathbf{n}$, $\mathbf{n}'$ defined in Theorem 13 and Lemma 14 are identical, steps 1,2, and 3 in the proof of Theorem 13 in Appendix D follow.

From step 1, we have $R_k = R'_k$ in distribution, and therefore the first two moments of R_k and R'_k are equal for each $k \in \mathbb{N}$. From step 2, it suffices to show that, for $\mathbf{n} = (n_1, \dots, n_m)$ and $\mathbf{n}' = (n_1, \dots, n_{t-1}, n_{t+1}, \dots, n_m)$ for some $t \in [m-1]$, we have $\bar{Z}(\mathbf{n}) \geqslant \bar{Z}(\mathbf{n}')$. From step 3, we can write the probability mass function for V'_k in terms of the probability mass function P_{V_k}, as in (11). Thus, the difference in the first moments of V_k and V'_k is given by

$$d_0 \triangleq \mathbb{E}[V_k - V'_k] = \sum_{i=t+1}^{m} P_{V_k}(i) = \frac{(F(n_m) - F(n_t))}{F(n_m)} \geqslant 0.$$

From (12), we recall that $d_1 = \mathbb{E}[n'_{V'_k} - n_{V_k}] = (n_{t+1} - n_t)P_{V_k}(t) \geqslant 0$. Furthermore, since $n'_{m'} = n_m$ and $R'_k = R_k$ in distribution, we gather that $\mathbb{E}[T'_k - T_k] = d_1$.

Using the expression for limiting average feedback rate in Lemma 6 for a fixed block assignment vector, we can write the difference between limiting empirical average feedback rates for two block assignment vectors $\mathbf{n}'$ and $\mathbf{n}$ as

$$\begin{aligned}\bar{Z}(\mathbf{n}') - \bar{Z}(\mathbf{n}) &\leqslant \frac{m'\mathbb{E}R'_k + \mathbb{E}V'_k}{\mathbb{E}T'_k} - \frac{m\mathbb{E}R_k + \mathbb{E}V_k}{\mathbb{E}T_k} \\ &= \frac{-\mathbb{E}T_k(\mathbb{E}R_k + d_0) - d_1(m\mathbb{E}R_k + \mathbb{E}V_k)}{\mathbb{E}T_k(\mathbb{E}T_k + d_1)} \leqslant 0.\end{aligned}$$

Appendix F
Proof of Lemma 15

This result follows from mathematical induction on the codeword length m. Hence, it suffices to show that, for block assignment vectors $\mathbf{n} = n_1 - 1 + [m]$ and $\mathbf{n}' = n_1 - 1 + [m+1]$, the corresponding feedback rates satisfy $\bar{Z}(\mathbf{n}') \geqslant \bar{Z}(\mathbf{n})$. For such block assignment vectors $\mathbf{n}$ and $\mathbf{n}'$, we use the notation R_k, V_k, T_k, m and R'_k, V'_k, T'_k, m', respectively, as defined in Notation 10. We observe that the block assignment vectors are contiguous in this case and differ only in codeword lengths, in contrast to the refinement considered in Lemma 14. That is, the incremental difference at each sub-block $i > 1$ is $\ell_{i+1} = n_{i+1} - n_i = 1$.

Since $m' = m + 1$, we can write the probability mass function for V'_k in terms of the probability mass function P_{V_k} as

$$P_{V'_k}(i) = \frac{P_{V_k}(i)F(n_m)\mathbb{1}_{\{i \in [m]\}}}{F(n_m + 1)} + \frac{F(n_m + 1) - F(n_m)}{F(n_m + 1)}\mathbb{1}_{\{i = m'\}}.$$

We will denote $P_{V'_k}(m+1)$ by $\alpha \in [0,1]$ in the following for simplicity. We can write

$$d_3 \triangleq \mathbb{E}[R_k - R'_k] = \frac{\alpha}{F(N)} = \alpha(1 + \mathbb{E}R_k) \geqslant 0.$$

From the probability mass function for V'_k, in terms of that for V_k, we can write the difference between the means of V'_k and V_k as

$$d_0 \triangleq \mathbb{E}[V'_k - V_k] = \alpha(m + 1 - \mathbb{E}V_k) \geqslant 0.$$

Similarly, we can also find the difference between the means of n_{V_k} and $n'_{V'_k}$ as

$$d_1 \triangleq \mathbb{E}[n'_{V'_k} - n_{V_k}] = (n_m + 1 - \mathbb{E}n_{V_k})\alpha \geqslant 0.$$

Recall that $\mathbb{E}T_k = n_m\mathbb{E}R_k + \mathbb{E}n_{V_k}$ and $T'_k = n'_{m'}\mathbb{E}R'_k + \mathbb{E}n'_{V'_k}$. Therefore, using the expressions for individual terms, we can write

$$\mathbb{E}[T'_k - T_k] = \mathbb{E}R_k - (n_m + 1)d_3 + d_1.$$

Using the expression for limiting average feedback rate in Lemma 6 for a fixed block assignment vector, we can write the difference between limiting empirical average feedback rate for two block assignment vectors $\mathbf{n}'$ and $\mathbf{n}$ as

$$\bar{Z}(\mathbf{n}') - \bar{Z}(\mathbf{n}) = \frac{m'\mathbb{E}R'_k + \mathbb{E}V'_k}{\mathbb{E}T'_k} - \frac{m\mathbb{E}R_k + \mathbb{E}V_k}{\mathbb{E}T_k}.$$

Substituting for $\mathbb{E}T'_k$ and $\mathbb{E}R'_k$ in the above equation from steps 1–3, and simplifying the above terms, we obtain

$$\bar{Z}(\mathbf{n}') - \bar{Z}(\mathbf{n}) = \frac{(1-\alpha)\mathbb{E}R_k((n_m - m)\mathbb{E}R_k + \mathbb{E}(n_{V_k} - V_k))}{\mathbb{E}T_k\mathbb{E}T'_k}.$$

Since $n_1 \geqslant 1$, we have $n_m - m = n_1 - 1 \geqslant 0$ and $n_{V_k} = n_1 - 1 + V_k \geqslant V_k$. Therefore, the difference between the feedback rates $\bar{Z}(\mathbf{n}') - \bar{Z}(\mathbf{n}) \geqslant 0$, and hence the desired result holds.

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