Lecture 6: Functions : Injectivity, Surjectivity, and Bijectivity

1 Injectivity, Surjectivity, Bijectivity

We are interested in finding out the conditions for a function to have a left inverse, or right inverse, or both.

Definition 1.1. Let $f\colon A \to B$ be a function.

1. The map f is **injective** (also called one-to-one/monic/into) if $x \neq y$ implies $f(x) \neq f(y)$ for all $x, y \in A$. Equivalently, $f(x) = f(y)$ implies $x = y$ for all $x, y \in A$.

2. The map f is **surjective** (onto/epic) if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$, equivalently $f(A) = B$.

3. The map f is **bijective** if it is both injective and surjective.

Lemma 1.2. *Let $f : A \to B$ be a function. Then the following are true.*

i) Function f is injective iff $f^{-1}(\{b\})$ has at most one element for all $b \in B$.

ii) Function f is surjective iff $f^{-1}(\{b\})$ has at least one element for all $b \in B$.

iii) Function f is bijective iff $f^{-1}(\{b\})$ has exactly one element for all $b \in B$.

Example 1.3. A function $f\colon \mathbb{R} \to \mathbb{R}$ on real line is a special function. This function is injective iff any horizontal line intersects at at most one point, surjective iff any horizontal line intersects at at least one point, and bijective iff any horizontal line intersects at exactly one point.

In the following lemma, we see that injectivity, surjectivity, and bijectivity is preserved by composition of functions.

Lemma 1.4. *Let $f\colon A \to B$, $g\colon B \to C$ be functions.*

i) Functions f, g are injective, then function $f \circ g$ injective.

ii) Functions f, g are surjective, then function $f \circ g$ surjective.

iii) Functions f, g are bijective, then function $f \circ g$ bijective.

In the following theorem, we show how these properties of a function are related to existence of inverses.

Theorem 1.5. *Let A and B be two non-empty sets and let $f\colon A \to B$ be a function.*

i) Function f has a right inverse iff f is surjective.

ii) Function f has a left inverse iff f is injective.

iii) Function f has a inverse iff f is bijective.

Proof. Let A and B be non-empty sets and $f : A \to B$ a function.

i) $\Rightarrow$. Suppose f has a right inverse g, then $f \circ g = 1_B$. We will show f is surjective. Let $b \in B$, we need to find an element $a \in A$ such that $f(a) = b$. Let $a = g(b)$ then $f(a) = (f \circ g)(b) = 1_B(b) = b$.
$\Leftarrow$. Suppose f is surjective. We wish to show that f has a right inverse, i.e., there exists a map $g : B \to A$ such that $f \circ g{=}1_B$. We define g as follows. For each $b \in B$, let $g(b) = a$ for some $a \in f^{-1}(\{b\})$. Notice that $f^{-1}(\{b\})$ is non-empty for all $b \in B$ since f is surjective. Now $(f \circ g)(b) = b$ for all $b \in B$ by definition. Hence $f \circ g = 1_B$.

ii) $\Rightarrow$. Suppose f has a left inverse h, then $h \circ f = 1_A$. Let $x, y \in A$ such that $f(x) = f(y) \in B$. By definition of left inverse we have then $x = (h \circ f)(x) = (h \circ f)(y) = y$. Hence, f is injective.
$\Leftarrow$. Suppose f is injective. We wish to show that f has a left inverse, i.e., there exists a map $h\colon B \to A$ such that $h \circ f{=}1_A$. We define $h : B \to A$ as follows. For each $b \in f(A)$, let $h(b) = f^{-1}(\{b\})$. Notice that $f^{-1}(\{b\})$ has exactly one element for all $b \in f(A)$ since f is injective. Now $(h \circ f)(a) = a$ for all $a \in A$ by definition. Hence $h \circ f = 1_A$.

iii) Follows from first two parts.

□

Theorem 1.6. *Let A and B be non empty sets and let $f\colon A \to B$ be a function.*

(i) the function f is surjective iff $g \circ f = h \circ f$ implies $g = h$ for all functions $g, h\colon B \to X$ for all sets X.

(ii) The function f is injective iff $f \circ g = f \circ h$ implies $g = h$ for all functions $g, h\colon Y \to A$ for all sets Y.

Proof. Let A and B be non empty sets and let $f\colon A \to B$ be a function.

1. $\Rightarrow$. Assume f surjective. Let $g, h\colon B \to X$ such that $g \circ f = h \circ f$ for some set X. By Theorem 1.5(i), function f has right inverse $q\colon B \to A$. By associativity, we have $(g \circ f) \circ q = g \circ (f \circ q)$ and thus $g \circ 1_B = h \circ 1_B$ and so $g = h$.
   $\Leftarrow$. We assume f not surjective. Let $b \in B$ such that $f^{-1}(\{b\}) = \emptyset$. Let $X = \{1, 2\}$ define $g, h\colon B \to X$ by $g(y) = 1$ for all $y \in B$ and $h(y) = 1$ for all $y \in B \setminus \{b\}$ and $h(b) = 2$. It is clear that $g \circ f = h \circ f$, even though $g \neq h$.

2. $\Rightarrow$. Assume f injective. Let $g, h\colon Y \to A$ such that $f \circ g = f \circ h$ for some set Y. By Theorem 1.5(ii), function f has left inverse $q\colon B \to A$. By associativity, we have $q \circ (f \circ g) = (q \circ f) \circ g$. Therefore, thus $1_A \circ g = 1_A \circ h$ and so $g = h$.
   $\Leftarrow$. We assume f not injective. Let $x \neq y \in A$ such that $f(x) = f(y)$. Let $Y = \{1, 2\}$ define $g, h\colon Y \to A$ by $g(1) = x, g(2) = y$. It is to see that $f \circ g = f \circ h$, even though $g \neq h$.

□

2 Sets of functions

Definition 2.1. Let A and B be sets. The set $\mathcal{F}(A, B)$ is defined to be the set of all functions $f\colon A \to B$. That is,

$$\mathcal{F}(A, B) = \{f\colon A \to B |\, f \text{ is a function }\}.$$

Remark 1. If A and B are not empty, then $\mathcal{F}(A, B)$ is non-empty.

Example 2.2. 1. Let $A = \{1, 2\}$ and $B = \{x, y\}$. Then, $\mathcal{F}(A, B) = \{(f, g, h, k)\}$, where $f = \{(1, x), (2, x)\}$, $g = \{(1, y), (2, y)\}$, $h = \{(1, x), (2, y)\}$, $k = \{(1, y), (2, x)\}$. These functions are illustrated in Figure 1 and Figure 2 respectively.

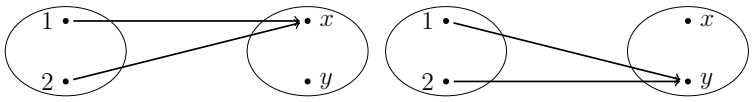

Figure 1: Functions $f = \{(1, x), (2, x)\}$, $g = \{(1, y), (2, y)\}$.

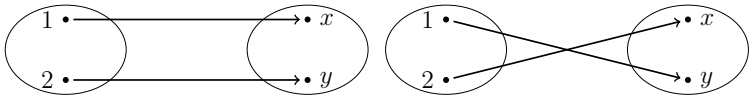

Figure 2: Functions $h = \{(1, x), (2, y)\}$, $k = \{(1, y), (2, x)$.

2. Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be set of real-valued functions on real line. Let $C(\mathbb{R}, \mathbb{R})$ and $D(\mathbb{R}, \mathbb{R})$ be set of real-valued continuous and derivable functions respectively. Then, $F(\mathbb{R}, \mathbb{R}) \supset C(\mathbb{R}, \mathbb{R}) \supset D(\mathbb{R}, \mathbb{R})$. In fact, let $k\colon D(\mathbb{R}, \mathbb{R}) \to F(\mathbb{R}, \mathbb{R})$ such that $k(f) = f'$. Then, k is not injective. For example, let $f(x) = x^2 + 5$ and $g(x) = x^2 + 7$ for all $x \in \mathbb{R}$. Then, $k(f) = k(g)$, but $f \neq g$.

3. Let $f \in \mathcal{F}(\mathbb{N}, \mathbb{R})$, then $f(1), f(2), \ldots$ is a sequences of real numbers. That is, $\mathcal{F}(\mathbb{N}, \mathbb{R})$ set of real-valued sequences.

Lemma 2.3. *Let A, B, C, D be sets suppose that there are bijective maps $f\colon A \to C$ and $g\colon B \to D$ then there is a bijective map between $\mathcal{F}(A, B)$ and $\mathcal{F}(C, D)$.*

Proof. Since f and g are bijective maps , They have inverse maps f^{-1} and g^{-1} respectively. Define $\Phi\colon \mathcal{F}(A, B) \to \mathcal{F}(C, D)$ by $\Phi(h) = g \circ h \circ f^{-1}$. for all $h \in \mathcal{F}(A, B)$, it's easy to see that $\Phi(h) \in \mathcal{F}(C, D)$ for all $h \in \mathcal{F}(A, B)$. We need to show Φ is bijective. One can see the corresponding commutative diagram in Figure 3.

Injective: Let $h, k \in \mathcal{F}(A, B)$ and suppose $\Phi(h) = \Phi(k)$. Then $g \circ h \circ f^{-1} = g \circ k \circ f^{-1}$. Hence, from associativity of composition of functions we have, $h = g^{-1} \circ (g \circ h \circ f^{-1}) \circ f = g^{-1} \circ (g \circ k \circ f^{-1}) \circ f = k$.

Surjective: Let $r \in \mathcal{F}(C, D)$, and define $t = g^{-1} \circ r \circ f$. Clearly, $t \in \mathcal{F}(A, B)$, and $\Phi(t) = g \circ (g^{-1} \circ r \circ f) \circ f^{-1} = r$. □

We can use set of function to study power sets. Indeed, we will show there is a bijection between these two in the following proposition.

Proposition 2.4. *Let A be a non empty set. Then, there is a bijective map from $F(A, \{0, 1\})$ to $\mathcal{P}(A)$.*

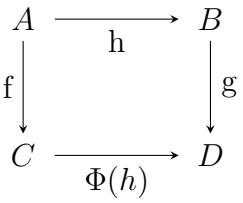

Figure 3: Commutative diagram for bijection between functional spaces $\mathcal{F}(A, B)$ and $\mathcal{F}(C, D)$, where $f : A \to C$ and $g : B \to D$ are bijective functions.

Proof. Let $\Phi\colon \mathcal{P}(A) \to \mathcal{F}(A, \{0, 1\})$ be defined as follows. Let $S \in \mathcal{P}(A)$, then $\Phi(S)\colon A \to \{0, 1\}$ is defined as

$$[\Phi(S)](x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S. \end{cases}$$

We will show that Φ is bijection in two different ways.

1. We can show that Φ is injective and surjective. **Injectivity:** Let $S, T \in \mathcal{P}(A)$ and suppose $\Phi(S) = \Phi(T)$, we will show $S = T$. Let $y \in S$, then $[\Phi(S)](y) = 1 = [\Phi(T)](y)$ then $y \in T$. Hence, we have shown $S \subseteq T$. Similarly, we can show, $T \subseteq S$.
   Surjectivity: Let $f \in \mathcal{F}(A, \{0, 1\})$. Define $S \in \mathcal{P}(A)$ such that $S = \{x \in A | f(x) = 1\} = f^{-1}(\{1\})$. We will show that $\Phi(S) = f$. For all $x \in S$, we have $\Phi(S)(x) = 1 = f(x)$. On the other hand, for all $x \notin S$, we have $\Phi(S)(x) = 0 = f(x)$.

2. We will construct an explicit inverse for Φ. To this end, we define $\Psi : \mathcal{F}(A, \{0, 1\}) \to \mathcal{P}(A)$ as

   $$\Psi(f) = f^{-1}(\{1\}) = \{x \in A : f(x) = 1\}, \text{ for all } f \text{ in } \mathcal{F}(A, \{0, 1\}).$$

   We will show $\Psi \circ \Phi = 1_{\mathcal{P}(A)}$ and $\Phi \circ \Psi = 1_{\mathcal{F}(A,\{0,1\})}$. Let $S \in \mathcal{P}(A)$, then $(\Psi \circ \Phi)(S) = [\Phi(S)]^{-1}(\{1\}) = S$. Now, let $f \in \mathcal{F}(A, \{0, 1\})$, such that $(\Phi \circ \Psi)(f) = \mathbb{1}_{\{f^{-1}(\{1\})\}}$. Clearly, $f(x) = \mathbb{1}_{\{f^{-1}(\{1\})\}}(x)$ for all $x \in A$.

□

3 Product of an indexed family of sets

We can define Cartesian product of two sets as $A \times B = \{(a, b), a \in A, b \in B\}$ in terms of $\{f \in \mathcal{F}(\{1, 2\}, A \cup B)$, such that $f(1) \in A$, and $f(2) \in B\}$.

Definition 3.1. Let I be a non empty set and let $\{A_i : i \in I\}$ be a family of sets indexed by I. The **product** of all the sets in this family of sets is defined as

$$\prod_{i \in I} A_i = \{f \in \mathcal{F}(I, \bigcup_{i \in I} A_i) : f(i) \in A_i, \text{ for all } i \in I\}.$$

If all the sets A_i are equal to a single set A. Then, we denote $\prod_{i \in A_i} A_i$ by A^I.

Proposition 3.2. *If $I \neq \emptyset$ and A is a set, then $A^I = f(I, A)$.*

Proof. Follows from the definition. □