A Systematic Approach to Incremental Redundancy over Erasure Channels

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ISIT

Random Coding + Hybrid ARQ

Consider the problem of communicating a k-bit message over a memoryless binary erasure channel (BEC) with erasure probability $0 \leq \epsilon < 1$, using random coding + hybrid ARQ*:

*ARQ: Automatic Repeat Request

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- Consider a random binary parity-check matrix H of size $(n-k) \times n$
- Consider an arbitrary mapping from k-bit messages to n-bit codewords in the null-space of matrix H

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- Consider a random binary parity-check matrix H of size $(n-k) \times n$
- Consider an arbitrary mapping from k-bit messages to n-bit codewords in the null-space of matrix H
- The source maps the message $\mathbf{x} = (x_1, \dots, x_k)$ to a codeword $\mathbf{c} = (c_1, \dots, c_n)$
- The source divides the codeword $\mathbf{c}$ into m sub-blocks $\mathbf{c}_1, \dots, \mathbf{c}_m$ for a given $2 \le m \le n$, where $\mathbf{c}_i = (c_{n_{i-1}}, \dots, c_{n_i})$ for $i \in [m] = \{1, \dots, m\}$, and $n_1, \dots, n_m$ are given integers such that $k \le n_1 < n_2 < \cdots < n_m = n$, and $n_0 = 0$

*ARQ: Automatic Repeat Request

Random Coding + Hybrid ARQ (Cont.)

- The source sends the first sub-block, $\mathbf{c}_1$
- The destination receives $\mathbf{c}_1$, or a proper subset thereof
- The destination performs ML decoding to recover the message $\mathbf{x}$, and depending on the outcome of decoding, sends an ACK or NACK to the source over a perfect feedback channel

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- If the source receives a NACK, it sends next sub-block, $\mathbf{c}_2$, and waits for an ACK or NACK again
- This action repeats until (i) the source receives an ACK; or (ii) it exhausts all the sub-blocks, and does not receive an ACK

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- This action repeats until (i) the source receives an ACK; or (ii) it exhausts all the sub-blocks, and does not receive an ACK

In case (i), the communication round succeeds, and the source starts a new communication round for the next message

In case (ii), the communication round fails, and the source starts a new communication round for the message $\mathbf{x}$.

Problem

Expected Effective Blocklength: The expected number of bits being sent by the source within a communication round
(the randomness comes from both the channel and the code)

Problem: To identify the aggregate sub-block sizes $n_1, \ldots, n_{m-1}$ such that the expected effective blocklength is minimized where a maximum of m sub-blocks (i.e., maximum m bits of feedback) are available in a communication round

Previous Works vs. This Work

Previous works (for channels other than BEC):

[1] Vakilinia-Williamson-Ranganathan-Divsalar-Wesel '14 (Feedback systems using non-binary LDPC codes with a limited number of transmissions, ITW)

[2] Williamson-Chen-Wesel '15 (Variable-length convolutional coding for short blocklengths with decision feedback, TCOM)

[3] Vakilinia-Ranganathan-Divsalar-Wesel '16 (Optimizing transmission lengths for limited feedback with non-binary LDPC examples, TCOM)

In this work, we propose a solution by extending the sequential differential optimization (SDO) framework of [3] for BEC

Expected Effective Blocklength

- R_t: the number of bits observed by the destination at time t, i.e., $R_t \sim B(t, 1-\epsilon)$
- P_{R_t}: the discrete probability measure associated with the random variable (r.v.) R_t, i.e.,

$$P_{R_t}(r) = \binom{t}{r}\epsilon^{t-r}(1-\epsilon)^r$$

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- $P_{\mathrm{s}}(r)$: the probability of decoding success given that the number of bits observed by the destination is r, i.e.,

$$P_{\mathrm{s}}(r) = \begin{cases} 0 & 0 \le r < k \\ \prod_{l=0}^{n-r-1} \left(1 - 2^{l-(n-k)}\right) & k \le r < n \\ 1 & r \ge n \end{cases}$$

Expected Effective Blocklength (Cont.)

- $P_{\mathrm{ACK}}(t)$: the probability that the destination sends an ACK to the source at time t or earlier, i.e.,

$$P_{\mathrm{ACK}}(t) = \begin{cases} 1 - \sum_{e=0}^{t} (1 - P_{\mathrm{s}}(t-e)) P_{R_t}(t-e) & k \le t \le n \\ 0 & 0 \le t < k \end{cases}$$

Expected Effective Blocklength (Cont.)

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- S: the index of last sub-block being sent by the source within a communication round
- $\mathbb{E}[n_S]$: the expected effective blocklength, i.e.,

$$\mathbb{E}[n_S] = n_m + \sum_{i=1}^{m-1}(n_i - n_{i+1})P_{\mathrm{ACK}}(n_i)$$

Problem: To identify $n_1, \ldots, n_{m-1}$ such that $\mathbb{E}[n_S]$ is minimized

Multi-Dimensional vs. One-Dimensional Optimization

Challenge: The problem of minimizing $\mathbb{E}[n_S]$ is a multi-dimensional optimization problem with integer variables $n_1, \ldots, n_{m-1}$

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Challenge: The problem of minimizing $\mathbb{E}[n_S]$ is a multi-dimensional optimization problem with integer variables $n_1, \ldots, n_{m-1}$

Idea: Sequential differential optimization (SDO) reduces the problem to a one-dimensional optimization with integer variable n_1

Recall

$$\mathbb{E}[n_S] = n_m + \sum_{i=1}^{m-1} (n_i - n_{i+1}) P_{\mathrm{ACK}}(n_i)$$

Suppose that a smooth approximation $F(t)$ of $P_{\mathrm{ACK}}(t)$ is given

Define

$$\tilde{\mathbb{E}}[n_S] = n_m + \sum_{i=1}^{m-1} (n_i - n_{i+1}) F(n_i)$$

Sequential Differential Optimization (SDO)

Recall

$$\tilde{\mathbb{E}}[n_S] = n_m + \sum_{i=1}^{m-1}(n_i - n_{i+1})F(n_i)$$

SDO: Given $\tilde{n}_1, \ldots, \tilde{n}_{i-1}$, an approximation $\tilde{n}_i$ of the optimal value of n_i for $2 \leq i \leq m-1$ can be computed via setting the partial derivative of $\tilde{\mathbb{E}}[n_S]$ with respect to n_{i-1} to zero and solving for n_i

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$\rightsquigarrow$ Given $\tilde{n}_1$ (and $\tilde{n}_0 = -\infty$), an approximation $\tilde{n}_i$ of the optimal value of n_i for all $2 \leq i \leq m-1$ can be obtained sequentially by

$$\tilde{n}_i = \tilde{n}_{i-1} + \left\lceil (F(\tilde{n}_{i-1}) - F(\tilde{n}_{i-2})) \left(\left. \frac{dF(t)}{dt} \right|_{t=\tilde{n}_{i-1}} \right)^{-1} \right\rceil$$

$\rightsquigarrow$ a one-dimensional optimization problem with variable n_1

Challenge: To find a smooth approximation $F(t)$ to $P_{\mathrm{ACK}}(t)$

Main Idea and Contributions

Fact: $P_{\mathrm{ACK}}(t)$ for $t < n$ matches the CDF of the r.v. N_n that represents the length of a communication round

Idea:

- To study the asymptotic behavior of the mean and variance of the r.v. N_n as n grows large, and
- To approximate $P_{\mathrm{ACK}}(t)$ by the CDF of a continuous r.v. with a mean and variance matching the mean and variance of the r.v. N_n as n grows large

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- To approximate $P_{\mathrm{ACK}}(t)$ by the CDF of a continuous r.v. with a mean and variance matching the mean and variance of the r.v. N_n as n grows large

In this work, we show that

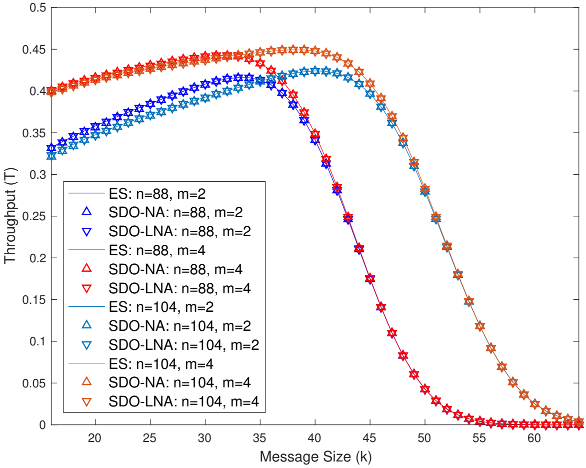
$$\lim_{n\to\infty} \mathbb{E}[N_n] = (k + c_0)/(1 - \epsilon)$$

and

$$\lim_{n\to\infty} \mathrm{Var}(N_n) = ((k + c_0)\epsilon + c_0 + c_1)/(1 - \epsilon)^2$$

where $c_0 = 1.60669...$ is the Erdös-Borwein constant, and $c_1 = 1.13733...$ is the digital search tree constant

Numerical Results



$\epsilon = 0.5$

$16 \leq k \leq 64$

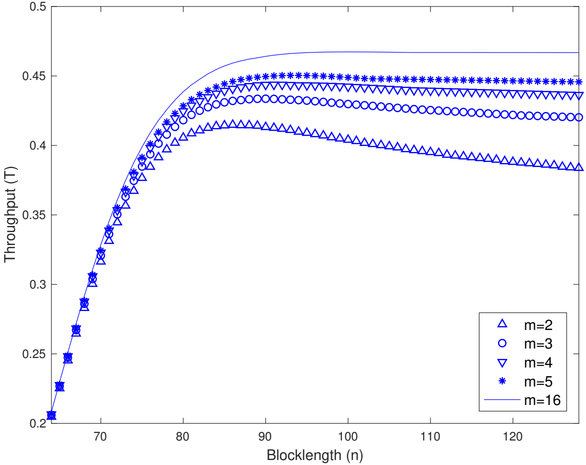
$n = 88, 104$

$m = 2, 4$

$$T = \frac{k P_{\mathrm{ACK}}(n)}{\mathbb{E}[n_S]}$$

- ES: Optimization by Exhaustive Search
- SDO-NA: SDO based on Normal Approximation
- SDO-LNA: SDO based on Log-Normal Approximation

Numerical Results (Cont.)



$\epsilon = 0.5$

$k = 32$

$64 \leq n \leq 128$

$m = 2, 3, 4, 5, 16$

$$T = \frac{kP_{\mathrm{ACK}}(n)}{\mathbb{E}[n_S]}$$

- the benefit in terms of throughout for $m \geq 5$ becomes relatively small
- a small number of sub-blocks (i.e., a few bits of feedback) suffice to achieve a throughput close to that obtained with unlimited feedback

Proof Steps

In this work, we show that

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Proof Steps:

- Analysis of the length of a communication round in the asymptotic regime over a lossless channel

  (by using closed-form formulas for several sums of products)
- Extension of the previous analysis for lossy channels

  (by showing matching lower and upper bounds)

Asymptotic Analysis over A Lossless Channel

Assume

- $\epsilon = 0$, i.e., the channel is lossless
- $m = n$, i.e., each sub-block is one bit

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Define

- M_n: the number of bits needed for the message to become decodable, following the prescribed order in the codeword
- P_{M_n}: the discrete probability measure for the r.v. M_n, i.e.,

$$P_{M_n}(r) = P_{\mathrm{s}}(r) - P_{\mathrm{s}}(r-1)$$

$$\rightsquigarrow \quad P_{M_n}(r) = \begin{cases} 2^{k-r} \prod_{l=0}^{n-r-1} \left(1 - 2^{l-(n-k)}\right) & k \le r \le n \\ 0 & \text{otherwise} \end{cases}$$

Goal: To study $\lim_{n\to\infty} \mathbb{E}[M_n]$ and $\lim_{n\to\infty} \mathrm{Var}(M_n)$

$\lim_{n\to\infty} \mathbb{E}[M_n]$ and $\lim_{n\to\infty} \mathrm{Var}(M_n)$

For any n,

$$\mathbb{E}[M_n] = \sum_{r=k}^{n} r P_{M_n}(r) = \sum_{i=0}^{n-k} (k+i) 2^{-i} \prod_{j=i+1}^{n-k} \left(1 - 2^{-j}\right)$$

and

$$\mathbb{E}[M_n^2] = \sum_{r=k}^{n} r^2 P_{M_n}(r) = \sum_{i=0}^{n-k} (k+i)^2 2^{-i} \prod_{j=i+1}^{n-k} \left(1 - 2^{-j}\right)$$

$\lim_{n\to\infty} \mathbb{E}[M_n]$ and $\lim_{n\to\infty} \mathrm{Var}(M_n)$

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Theorem

For any k, $\lim_{n\to\infty} \mathbb{E}[M_n] = k + c_0$ *and* $\lim_{n\to\infty} \mathrm{Var}(M_n) = c_0 + c_1$ *where* $c_0 = \sum_{i=1}^{\infty} \frac{1}{2^i - 1} = 1.60669\ldots$ *is the Erdös-Borwein constant, and* $c_1 = \sum_{i=1}^{\infty} \frac{1}{(2^i-1)^2} = 1.13733\ldots$ *is the digital search tree constant*

Proof: By using the closed-form formulas for several infinite sums of infinite products

Asymptotic Analysis over A Lossy Channel

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Define

- E_r: the number of bits erased before r bits are observed by the destination, i.e., $E_r \sim NB(r, \epsilon)$
- P_{E_r}: the discrete probability measure for the r.v. E_r, i.e.,

$$P_{E_r}(e) = \binom{r+e-1}{e} \epsilon^e (1-\epsilon)^r$$

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- N_n: the length of a communication round
- P_{N_n}: the discrete probability measure for the r.v. N_n, i.e.,

$$P_{N_n}(t) = \begin{cases} \sum_{r=k}^{t} P_{E_r}(t-r) P_{M_n}(r) & k \leq t < n \\ \sum_{u=n}^{\infty} \sum_{r=k}^{u} P_{E_r}(u-r) P_{M_n}(r) & t = n \end{cases}$$

Goal: To study $\lim_{n\to\infty} \mathbb{E}[N_n]$ and $\lim_{n\to\infty} \mathrm{Var}(N_n)$

$\lim_{n\to\infty} \mathbb{E}[N_n]$ and $\lim_{n\to\infty} \mathrm{Var}(N_n)$

- $\rightsquigarrow$ $N_n < n$: the destination can recover the message before all the codeword bits are sent by the source
- $\rightsquigarrow$ $N_n = n$: all the codeword bits are exhausted by the source, and the destination may or may not recover the message

For any n,

$$\mathbb{E}[N_n] = \sum_{r=k}^{n} \sum_{e=0}^{\infty} \min(r+e, n) P_{E_r}(e) P_{M_n}(r)$$

and

$$\mathbb{E}[N_n^2] = \sum_{r=k}^{n} \sum_{e=0}^{\infty} \min((r+e)^2, n^2) P_{E_r}(e) P_{M_n}(r).$$

$\lim_{n\to\infty} \mathbb{E}[N_n]$ and $\lim_{n\to\infty} \mathrm{Var}(N_n)$

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$$\mathbb{E}[N_n^2] = \sum_{r=k}^{n} \sum_{e=0}^{\infty} \min((r+e)^2, n^2) P_{E_r}(e) P_{M_n}(r).$$

Theorem

For any k and ϵ, $\lim_{n\to\infty} \mathbb{E}[N_n] = \mu(k,\epsilon) \triangleq \frac{k+c_0}{1-\epsilon}$ and $\lim_{n\to\infty} \mathrm{Var}(N_n) = \sigma^2(k,\epsilon) \triangleq \frac{(k+c_0)\epsilon + c_0 + c_1}{(1-\epsilon)^2}$

Proof: By showing matching lower and upper bounds

An Upper Bound on $\lim_{n\to\infty} \mathbb{E}[N_n]$

Since $\min(r+e,n) \le r+e$,

$$\begin{aligned}\mathbb{E}[N_n] &= \textstyle\sum_{r=k}^{n} P_{M_n}(r) \sum_{e=0}^{\infty} \min(r+e,n) P_{E_r}(e) \\ &\le \textstyle\sum_{r=k}^{n} P_{M_n}(r) \sum_{e=0}^{\infty} (r+e) P_{E_r}(e)\end{aligned}$$

for all n.

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for all n.

Since $E_r \sim NB(r, \epsilon)$,

$$\begin{aligned}\textstyle\sum_{e=0}^{\infty}(r+e)P_{E_r}(e) &= r\textstyle\sum_{e=0}^{\infty} P_{E_r}(e) + \sum_{e=0}^{\infty} eP_{E_r}(e) \\ &= r + \mathbb{E}[E_r] = r/(1-\epsilon)\end{aligned}$$

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Thus,

$$\mathbb{E}[N_n] \le \sum_{r=k}^{n} r P_{M_n}(r)/(1-\epsilon) = \mathbb{E}[M_n]/(1-\epsilon)$$

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Thus,

$$\mathbb{E}[N_n] \le \textstyle\sum_{r=k}^{n} rP_{M_n}(r)/(1-\epsilon) = \mathbb{E}[M_n]/(1-\epsilon)$$

for all n.

Since $\lim_{n\to\infty} \mathbb{E}[M_n] = k + c_0$ (by the result of the lossless case),

$$\lim_{n\to\infty} \mathbb{E}[N_n] \le (k+c_0)/(1-\epsilon)$$

A Lower Bound on $\lim_{n\to\infty} \mathbb{E}[N_n]$

Since $\min(r+e, n) = r + e$ for $0 \le e \le n - r$,

$$\begin{aligned}\mathbb{E}[N_n] &= \textstyle\sum_{r=k}^{n} \sum_{e=0}^{\infty} \min(r+e, n) P_{E_r}(e) P_{M_n}(r) \\ &\ge \textstyle\sum_{r=k}^{n} \sum_{e=0}^{n-r} (r+e) P_{E_r}(e) P_{M_n}(r)\end{aligned}$$

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for all n.

Since $P_{M_n}(r)$ is monotone decreasing in n for all $k \le r \le n$,

$$P_{M_n}(r) \ge \lim_{n\to\infty} P_{M_n}(r) = 2^{k-r} \textstyle\prod_{j=r-k+1}^{\infty} (1-2^{-j})$$

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Thus,

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for all n.

Since by the closed-form formulas for several sums of products,

$$\sum_{r=k}^{\infty} 2^{k-r} \sum_{e=0}^{\infty} (r+e) P_{E_r}(e) \prod_{j=r-k+1}^{\infty} (1-2^{-j}) = (k+c_0)/(1-\epsilon)$$

then,

$$\lim_{n\to\infty} \mathbb{E}[N_n] \ge (k+c_0)/(1-\epsilon)$$

Summary and Ongoing Work

In this work:

- Considered the problem of communicating a message over a BEC, using random coding + hybrid ARQ
- Proposed a framework based on the sequential differential optimization (SDO) to optimize the parameters of the system such that the average throughput of the system is maximized

Ongoing work: Extending the proposed SDO-based framework

- for scenarios with constrained feedback rate
- for channels with memory