Tracking an AR(1) Process with limited communication

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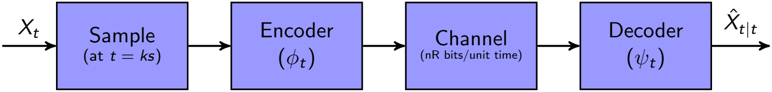
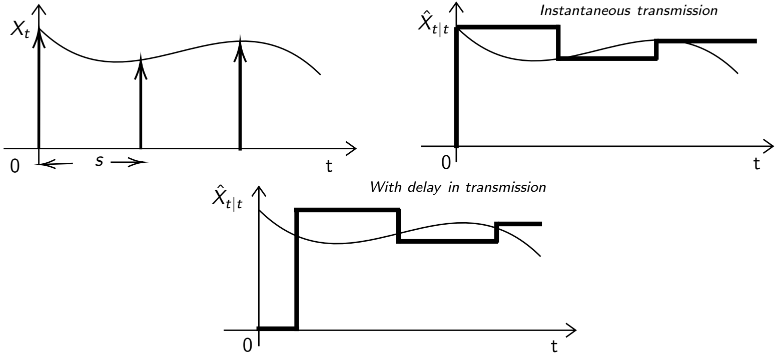
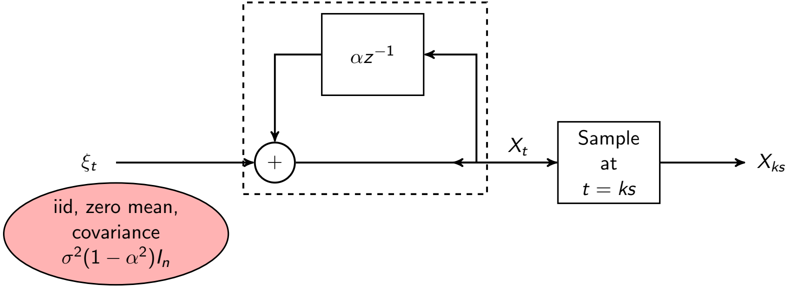
Remote real-time tracking



$\phi_t : \mathcal{X}^{k+1} \to \{0,1\}^{nsR}$ $\qquad$ $\psi_t : \{0,1\}^{nR(t-1)} \to \mathcal{X}$



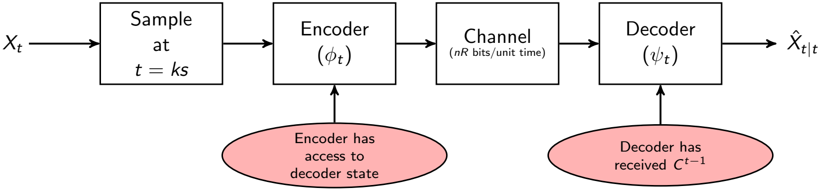
Source Process



n-dimensional discrete source process

- AR(1) process: $X_t = \alpha X_{t-1} + \xi_t$ for all $t \geq 0$
- $\sup_{t \in \mathbb{Z}^+} \frac{1}{n}\sqrt{\mathbb{E}\|X_t\|_2^4}$ is bounded

Problem description



Problem Statement

- Instantaneous tracking error $D_t(\phi, \psi, X) \triangleq \frac{1}{n}\mathbb{E}\|X_t - \hat{X}_{t|t}\|_2^2$.
- Optimum asymptotic maxmin tracking accuracy,

$$\delta^*(R, s, \mathbb{X}) = \lim_{T\to\infty} \lim_{n\to\infty} \left[\sup_{(\phi,\psi)} \inf_{X\in\mathbb{X}_n} 1 - \frac{\frac{1}{T}\sum_{t=0}^{T-1} D_t(\phi, \psi, X)}{\sigma^2} \right]$$

- Design (ϕ, ψ) that attains $\delta^*(R, s, \mathbb{X})$

Existing Works

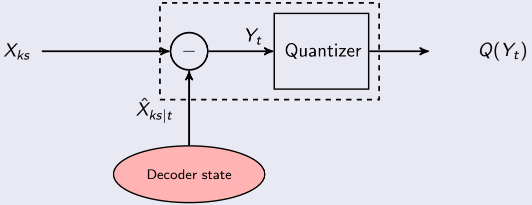
- Structural results on real time encoders
  - Witsenhausen(1979), Teneketzis(2006), Linder and Yuksel(2017) etc.
- Remote estimation under communication constraints
  - Wong and Brockett(1997), Nair and Evans(1997), Nayyar and Basar(2013), Chakravorthy and Mahajan(2017), Sun and Polyanskiy(2017) etc.
- Encoding stationary sources with noisy/noiseless rate limited samples
  - Zamir and Feder(1995), Zamir(2012), Kipnis et. al.(2015),
- Sequential coding for correlated sources
  - Viswanathan(2000), Khina et.al.(2017)

Current setting

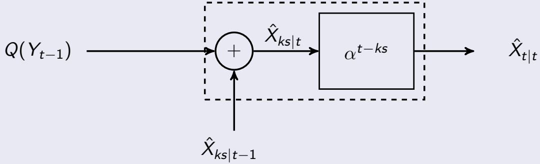
- Real-time estimation of AR(1) process
- Rate-limited channel with unit delay per channel use

Achievability Scheme

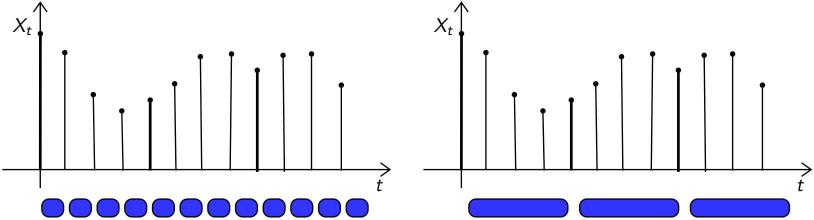
Encoder Structure



Decoder Structure



Encoder strategy: Fast or Precise?



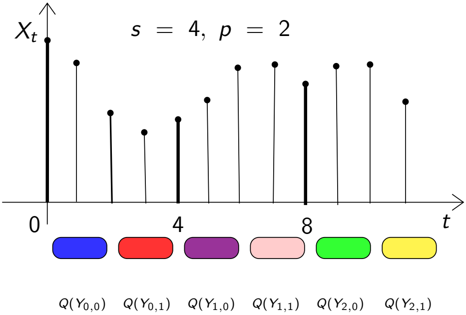
Fast but loose **Slow and Precise**

Optimal update strategy

What is the encoding strategy for an AR(1) process under periodic sampling that maximizes real-time tracking accuracy?

p-Successive Update Scheme

- Refine the estimate of the latest sample in every p time slots



- At $t = ks + jp$, encode $Y_{k,j} = X_{ks} - \hat{X}_{ks|ks+jp}$.
  e.g. $Y_{0,0} = X_0 - \hat{X}_{0|0}$, $Y_{0,1} = X_0 - \hat{X}_{0|2}$, $Y_{1,0} = X_4 - \hat{X}_{4|4}$ $\cdots$

(θ, ε)-quantizer

Definition

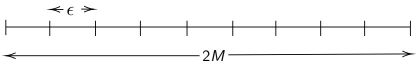
Fix $0 < M < \infty$. A quantizer $Q : \mathbb{R}^n \to \{0,1\}^{nR}$ constitutes an nR bit (θ, ε)-*quantizer* if for every vector $y \in \mathbb{R}^n$ such that $\frac{1}{n}\|y\|_2 \leq M$, we have

$$\mathbb{E}\|y - Q(y)\|_2^2 \leq \|y\|_2^2 \theta(R) + n\varepsilon^2.$$

for $0 \leq \theta \leq 1$ and $0 \leq \varepsilon$.

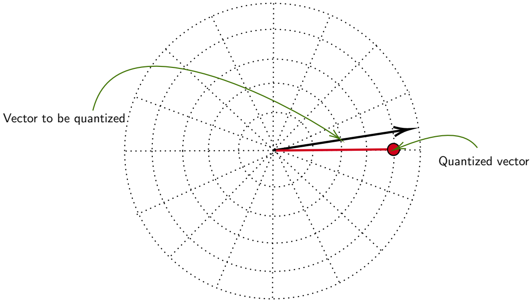
- e.g. a uniform quantizer with range $(-M, M)$, quantizing y, $|y| < M$
- The quantizer parameters : $\theta = 0$, $\epsilon^2 = M^2 2^{-2R}$

$$R = \log\lceil 2M/\epsilon \rceil$$

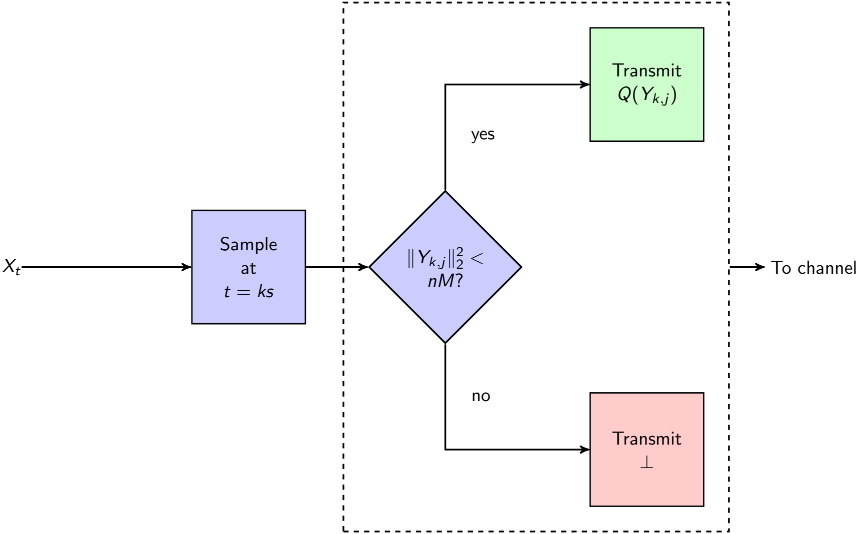


A gain shape quantizer

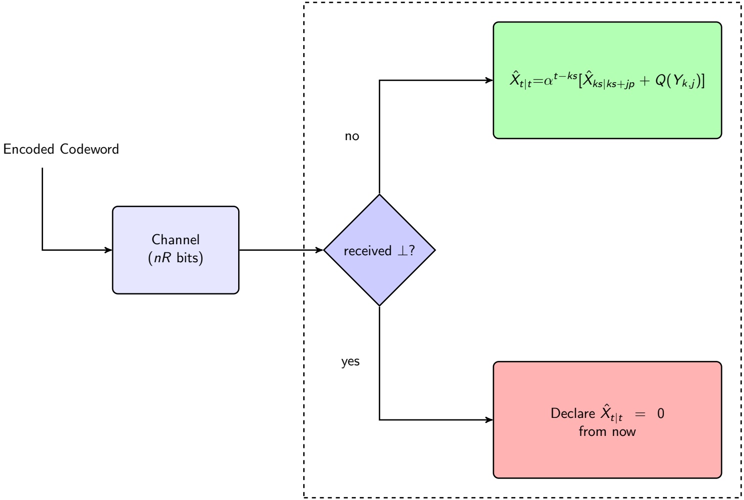
- To attain optimality, we need an ideal quantizer with $\theta(R) = 2^{-2R}$ and $\epsilon = 0$
- If Y is gaussian, use a gaussian codebook
- We use a random codebook based vector quantizer that quantizes the norm and the angle of a vector separately



Encoder at time $t = ks + jp$



Decoder at time $t = ks + (j+1)p + i$



Performance of p-Successive Update Scheme

Lemma

For $t = ks + jp + i$, the p-SU scheme employing a nRp bit (θ, ϵ) quantizer satisfies

$$D_t \leqslant \alpha^{2(t-ks)}\theta(Rp)^j D_{ks} + \sigma^2(1 - \alpha^{2(t-ks)}) + f(\epsilon, \beta).$$

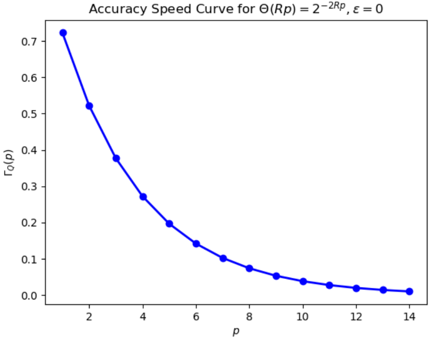
β : Upperbound on the probability of encoder failure

Guideline for choosing p

- Accuracy-speed curve for a (θ, ε)-quantizer,

$$\Gamma_Q(p) = \frac{\alpha^{2p}}{1 - \alpha^{2p}\,\theta(Rp)}\left(1 - \frac{\epsilon^2}{\sigma^2} - \theta(Rp)\right)$$

- For large T and negligible β, choose the p that maximizes accuracy-speed curve



Main results

Achievability: Lower bound for maxmin tracking accuracy

For $R > 0$ and $s \in \mathbb{N}$, the asymptotic maxmin tracking accuracy is bounded below as

$$\delta^*(R, s, \mathbb{X}) \geqslant \delta_0(R) g(s).$$

for $\delta_0(R) \triangleq \frac{\alpha^2(1-2^{-2R})}{(1-\alpha^2 2^{-2R})}$ *and* $g(s) \triangleq \frac{(1-\alpha^{2s})}{s(1-\alpha^2)}$ *for all* $s > 0$.

This bound is achieved using p-successive update scheme for $p = 1$ and a given realisation of (θ, ϵ) quantizer.

Main results

Achievability: Lower bound for maxmin tracking accuracy

For $R > 0$ and $s \in \mathbb{N}$, the asymptotic maxmin tracking accuracy is bounded below as

$$\delta^*(R, s, \mathbb{X}) \geqslant \delta_0(R)g(s).$$

Converse: Upper bound for maxmin tracking accuracy

For $R > 0$ and $s \in \mathbb{N}$, the asymptotic maxmin tracking accuracy is bounded above as

$$\delta^*(R, s, \mathbb{X}) \leqslant \delta_0(R)g(s).$$

The upper bound is obtained by considering a Gauss-Markov Process.

Conclusion

- Studied the real time estimation of AR(1) process under communication constraints
- An information theoretic upper bound for maxmin tracking accuracy for a fixed rate and sampling frequency
- For a fixed rate, high dimensional setting, the strategy of being *fast but loose* is universally optimal
- Outlined the performance requirements of the quantizer needed for achieving the optimal performance
- For non-asymptotic regime, the optimal strategy might differ