1. Consider a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in \mathbb{R}_+) \subseteq \mathcal{F}$ defined on this probability space. Let $\tau_n$ be a stopping time adapted to this filtration $\mathcal{F}_\bullet$ for each $n \in \mathbb{N}$. Let $\tau_n \leqslant \tau_{n+1}$ for all $n \in \mathbb{N}$ and $\tau_n \uparrow \tau$ almost surely, then show that the limit $\tau$ is a stopping time. A stopping time for which for which such a monotonically increasing sequence exists is called a *predictable stopping time*.

2. Let $\tau : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be a sequence of predictable stopping times. Show that $\tau_\infty \triangleq \sup_n \tau_n$ is a predictable stopping time.

3. Consider a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in \mathbb{R}_+) \subseteq \mathcal{F}$ defined on this probability space. Let $\tau_n$ be a stopping time adapted to this filtration $\mathcal{F}_\bullet$ for each $n \in \mathbb{N}$. Show that the following are also stopping times.
   (a) $\sup_n \tau_n$.
   (b) $\inf_n \tau_n$.
   (c) $\limsup_n \tau_n$.
   (d) $\liminf_n \tau_n$.

4. Consider a probability space $(\Omega, \mathcal{F}, P)$ and let $\sigma, \tau$ be stopping times on the natural filtration $\mathcal{F}_\bullet$, for $T = \mathbb{N}$. Show that the following events are in $\mathcal{F}_\tau$ and $\mathcal{F}_\sigma$.
   (a) $\{\sigma < \tau\}$
   (b) $\{\sigma = \tau\}$
   (c) $\{\sigma \leqslant \tau\}$

5. Consider a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in \mathbb{R}_+) \subseteq \mathcal{F}$ defined on this probability space. Let $\tau_1, \tau_2$ be stopping times adapted to this filtration $\mathcal{F}_\bullet$, then show that $\mathcal{F}_{\tau_1} \cap \mathcal{F}_{\tau_2} = \mathcal{F}_{\tau_1 \wedge \tau_2}$.

6. Consider a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in \mathbb{R}_+) \subseteq \mathcal{F}$ defined on this probability space. Let $\tau_n$ be a stopping time adapted to this filtration $\mathcal{F}_\bullet$ for each $n \in \mathbb{N}$. Let $\tau_n \geqslant \tau_{n+1}$ for all $n \in \mathbb{N}$ and $\tau_n \downarrow \tau$ almost surely, then show that $\mathcal{F}_\tau = \cap_n \mathcal{F}_{\tau_n}$.

7. Consider a probability space $(\Omega, \mathcal{F}, P)$. Show that for a stopping time $\tau$, and $\{B(t) : t \geqslant 0\}$, a standard Brownian motion, the Brownian motion reflected at $\tau$, $\{B^*(t) : t \geqslant 0\}$ defined as
$$B^*(t) = B(t)1_{\{t \leqslant \tau\}} + (2B(\tau) - B(t))1_{\{t > \tau\}}$$
is also a standard Brownian motion.

   **Hint:** Make use of the fact that Brownian motion is symmetric, i.e. $P(B(t) \leqslant x) = P(B(t) \geqslant (-x))$.

8. Consider a probability space $(\Omega, \mathcal{F}, P)$. On this, consider a Brownian motion $\{B(t) : t \geqslant 0\}$ such that $B(0) = 0$. Define the running maximum $M(t) = \max_{0 \leqslant s \leqslant t} B(t)$. For $a > 0$, show that $P\{M(t) > a\} = 2P\{B(t) > a\}$.
   **Hint:** Use the reflection principle from the previous problem.