

# Lecture-04: Stochastic Processes

## 1 Stochastic Processes

**Definition 1.1.** For an arbitrary index set  $T$  and a real valued function  $x \in \mathbb{R}^T$ , the projection operator  $\pi_t : \mathbb{R}^T \rightarrow \mathbb{R}$  is defined as  $\pi_t(x) \triangleq x_t$  for any  $x \in \mathbb{R}^T$ .

*Remark 1.* Recall that  $\pi_t^{-1}(-\infty, x] = \times_{s \in T} (-\infty, x_s]$  where  $x_s = x$  for  $s = t$  and  $x_s = \infty$  for all  $s \neq t$ . That is, we can write  $\pi_t^{-1}(-\infty, x_t] = (-\infty, x_t] \times_{s \in T: s \neq t} \mathbb{R}$  for any  $x_t \in \mathbb{R}$ .

**Definition 1.2 (Random process).** Let  $(\Omega, \mathcal{F}, P)$  be a probability space. For an arbitrary index set  $T$  and state space  $\mathcal{X} \subseteq \mathbb{R}$ , a map  $X : \Omega \rightarrow \mathcal{X}^T$  is called a **random process** if the projections  $X_t : \Omega \rightarrow \mathcal{X}$  defined by  $X_t(\omega) \triangleq (\pi_t \circ X)(\omega)$  are random variables on the given probability space. For each outcome  $\omega \in \Omega$ , we have a function  $X(\omega) \in \mathcal{X}^T$  called the **sample path** or the **sample function** of the process  $X$ .

*Remark 2.* A random process  $X$  defined on probability space  $(\Omega, \mathcal{F}, P)$  with index set  $T$  and state space  $\mathcal{X} \subseteq \mathbb{R}$ , can be thought of as

- (a) a map  $X : \Omega \times T \rightarrow \mathcal{X}$ ,
- (b) a map  $X : T \rightarrow \mathcal{X}^\Omega$ , i.e. a collection of random variables  $X_t : \Omega \rightarrow \mathcal{X}$  for each time  $t \in T$ ,
- (c) a map  $X : \Omega \rightarrow \mathcal{X}^T$ , i.e. a collection of sample functions  $X(\omega) \in \mathcal{X}^T$  for each random outcome  $\omega \in \Omega$ .

### 1.1 Classification

State space  $\mathcal{X}$  can be countable or uncountable, corresponding to discrete or continuous valued process. If the index set  $T$  is countable, the stochastic process is called **discrete-time** stochastic process or random sequence. When the index set  $T$  is uncountable, it is called **continuous-time** stochastic process. The index set  $T$  doesn't have to be time, if the index set is space, and then the stochastic process is spatial process. When  $T = \mathbb{R}^n \times [0, \infty)$ , stochastic process  $X$  is a spatio-temporal process.

### 1.2 Measurability

For random process  $X : \Omega \rightarrow \mathcal{X}^T$  defined on the probability space  $(\Omega, \mathcal{F}, P)$ , the projections  $X_t \triangleq \pi_t \circ X$  are  $\mathcal{F}$ -measurable random variables. Therefore, the set of outcomes  $A_{X_t}(x) \triangleq X_t^{-1}(-\infty, x] \in \mathcal{F}$  for all  $t \in T$  and  $x \in \mathbb{R}$ .

**Definition 1.3.** A random map  $X : \Omega \rightarrow \mathcal{X}^T$  is called  **$\mathcal{F}$ -measurable** and hence a random process, if the set of outcomes  $A_{X_t}(x_t) \triangleq X_t^{-1}(-\infty, x_t] \in \mathcal{F}$  for all  $t \in T$  and  $x_t \in \mathbb{R}$ .

**Definition 1.4.** The **event space generated by a random process**  $X : \Omega \rightarrow \mathcal{X}^T$  defined on a probability space  $(\Omega, \mathcal{F}, P)$  is given by

$$\sigma(X) \triangleq \sigma(A_{X_t}(x) : t \in T, x \in \mathbb{R}).$$

**Definition 1.5.** For a random process  $X : \Omega \rightarrow \mathcal{X}^T$  defined on the probability space  $(\Omega, \mathcal{F}, P)$ , we define the projection of  $X$  onto components  $S \subseteq T$  as the random vector  $X_S : \Omega \rightarrow \mathcal{X}^S$ , where  $X_S \triangleq (X_s : s \in S)$ .

*Remark 3.* The  $\mathcal{F}$ -measurability of process  $X$  implies that for any countable set  $S \subseteq T$ , we have  $A_{X_S}(x_S) \triangleq \cap_{s \in S} A_{X_s}(x_s) \in \mathcal{F}$  for  $x_S \in \mathcal{X}^S$ .

**Definition 1.6.** We can define  $A_X(x) \triangleq \cap_{t \in T} A_{X_t}(x_t) = \{\omega \in \Omega : (\pi_t \circ X)(\omega) \leq x_t\}$  for any  $x \in \mathbb{R}^T$ .

*Remark 4.* However,  $A_X(x)$  is guaranteed to be an event only when  $S \triangleq \{t \in T : \pi_t(x) < \infty\}$  is a countable set. In this case,

$$A_X(x) = \cap_{t \in T} A_{X_t}(x_t) = \cap_{s \in S} A_{X_s}(x_s) = A_{X_S}(x_S) \in \mathcal{F}.$$

*Remark 5.* For any finite subset  $S \subseteq T$  and real vector  $x \in \mathbb{R}^T$  such that  $x_t = \infty$  for any  $t \notin S$ , we define a set

$$B(x) \triangleq \left\{ y \in \mathbb{R}^T : y_t \leq x_t \right\} = \bigtimes_{t \in T} (-\infty, x_t] = \bigtimes_{s \in S} (-\infty, x_s] \bigtimes_{t \notin S} \mathbb{R} = \bigcap_{t \in T} \pi_t^{-1}(-\infty, x_t].$$

The measurability of the random process  $X$  implies that for any such set  $B(x)$ , we have

$$A_X(x) = X^{-1}(B(x)) = \bigcap_{t \in T} (X^{-1} \circ \pi_t^{-1})(-\infty, x_t] = \bigcap_{t \in T} X_t^{-1}(-\infty, x_t] = \bigcap_{s \in S} X_s^{-1}(-\infty, x_s] \in \mathcal{F}.$$

**Example 1.7 (Bernoulli sequence).** Consider a sample space  $\{H, T\}^{\mathbb{N}}$ . We define a mapping  $X : \Omega \rightarrow \{0, 1\}^{\mathbb{N}}$  such that  $X_n(\omega) = \mathbb{1}_{\{H\}}(\omega_n) = \mathbb{1}_{\{\omega_n=H\}}$ . The map  $X$  is an  $\mathcal{F}$ -measurable random sequence, if each  $X_n : \Omega \rightarrow \{0, 1\}$  is a bi-variate  $\mathcal{F}$ -measurable random variable on the probability space  $(\Omega, \mathcal{F}, P)$ . Therefore, the event space  $\mathcal{F}$  must contain the event space generated by events  $E_n \triangleq \{\omega \in \Omega : X_n(\omega) = 1\} = \{\omega \in \Omega : \omega_n = H\} \in \mathcal{F}$ . That is,

$$\sigma(X) = \sigma(E_n : n \in \mathbb{N}).$$

### 1.3 Distribution

**Definition 1.8.** For a random process  $X : \Omega \rightarrow \mathcal{X}^T$  defined on the probability space  $(\Omega, \mathcal{F}, P)$ , we define a **finite dimensional distribution**  $F_{X_S} : \mathbb{R}^S \rightarrow [0, 1]$  for a finite  $S \subseteq T$  by

$$F_{X_S}(x_S) \triangleq P(A_{X_S}(x_S)) = P(\bigcap_{s \in S} A_{X_s}(x_s)), \quad x_S \in \mathbb{R}^S.$$

**Example 1.9.** Consider a probability space  $(\Omega, \mathcal{F}, P)$  defined by the sample space  $\Omega = \{H, T\}^{\mathbb{N}}$ , the event space  $\mathcal{F} \triangleq \sigma(E_n : n \in \mathbb{N})$  where  $E_n = \{\omega \in \Omega : \omega_n = H\}$ , and the probability measure  $P : \mathcal{F} \rightarrow [0, 1]$  defined by

$$P(\bigcap_{i \in F} E_i) = p^{|F|}, \text{ for all finite } F \subseteq \mathbb{N}.$$

Let  $X : \Omega \rightarrow \{0, 1\}^{\mathbb{N}}$  defined as  $X_n(\omega) = \mathbb{1}_{E_n}(\omega)$  for all outcomes  $\omega \in \Omega$  and  $n \in \mathbb{N}$ . For this random sequence, we can obtain the finite dimensional distribution  $F_{X_S} : \mathbb{R}^S \rightarrow [0, 1]$  for any finite  $S \subseteq T$  and  $x \in \mathbb{R}^S$  in terms of  $U \triangleq \{i \in S : x_i < 0\}$  and  $V \triangleq \{i \in S : x_i \in [0, 1]\}$ , as

$$F_{X_S}(x) = \begin{cases} 1, & U \cup V = \emptyset, \\ (1-p)^{|V|}, & U = \emptyset, V \neq \emptyset, \\ 0, & U \neq \emptyset. \end{cases} \quad (1)$$

To define a measure on a random process, we can either put a measure on sample paths ( $X(\omega) \in \mathcal{X}^T : \omega \in \Omega$ ), or equip the collection of random variables ( $X_t \in \mathcal{X} : t \in T$ ) with a joint measure. Either way, we are interested in identifying the joint distribution  $F : \mathbb{R}^T \rightarrow [0, 1]$ . To this end, for any  $x \in \mathbb{R}^T$ , we need to know

$$F_X(x) \triangleq P\left(\bigcap_{t \in T} \{\omega \in \Omega : X_t(\omega) \leq x_t\}\right) = P\left(\bigcap_{t \in T} X_t^{-1}(-\infty, x_t]\right) = P \circ X^{-1} \bigtimes_{t \in T} (-\infty, x_t].$$

First of all, we don't know whether  $A_X(x)$  is an event when  $T$  is uncountable. Though, we can verify that  $A_X(x) \in \mathcal{F}$  for  $x \in \mathbb{R}^T$  such that  $\{t \in T : x_t < \infty\}$  is countable. Second, even for a simple independent process with countably infinite  $T$ , any function of the above form would be zero if  $x_t$  is finite for all  $t \in T$ . That is, for any finite set  $S \subseteq T$ , we focus on the events  $A_S(x_S)$  and their probabilities. However, these are precisely the finite dimensional distributions. Set of all finite dimensional distributions of a stochastic process  $X : \Omega \rightarrow \mathcal{X}^T$  characterizes its distribution completely. Simpler characterizations of a stochastic process  $X$  are in terms of its moments. That is, the first moment such as mean, and the second moment such as correlations and covariance functions.

$$m_X(t) \triangleq \mathbb{E}X_t, \quad R_X(t, s) \triangleq \mathbb{E}X_t X_s, \quad C_X(t, s) \triangleq \mathbb{E}(X_t - m_X(t))(X_s - m_X(s)).$$

**Example 1.10.** Consider a probability space  $(\Omega, \mathcal{F}, P)$  defined by the sample space  $\Omega = \{H, T\}^{\mathbb{N}}$  and the event space  $\mathcal{F} \triangleq \sigma(E_n : n \in \mathbb{N})$  where  $E_n = \{\omega \in \Omega : \omega_n = H\}$ . Let  $X : \Omega \rightarrow \{0, 1\}^{\mathbb{N}}$  defined as  $X_n(\omega) = \mathbb{1}_{E_n}(\omega)$  for all outcomes  $\omega \in \Omega$  and  $n \in \mathbb{N}$ . For this random sequence, if we are given the finite dimensional distribution  $F_{X_S} : \mathbb{R}^S \rightarrow [0, 1]$  for any finite  $S \subseteq \mathbb{N}$  and  $x \in \mathbb{R}^S$  in terms of  $U \triangleq \{i \in S : x_i < 0\}$  and  $V \triangleq \{i \in S : x_i \in [0, 1)\}$ , as defined in Eq. (1). Then, we can find the probability measure  $P : \mathcal{F} \rightarrow [0, 1]$  is given by

$$P(\cap_{i \in F} E_i) = p^{|F|}, \text{ for all finite } F \subseteq \mathbb{N}.$$

Let  $q \triangleq (1 - p)$ , then the probability of observing  $m$  heads and  $r$  tails is given by  $p^m q^r$ . We can easily compute the mean, the auto-correlation, and the auto-covariance functions for this independent Bernoulli process

$$m_X(n) = \mathbb{E}X_n = p, \quad R_X(m, n) = \mathbb{E}X_m X_n = \mathbb{E}X_m \mathbb{E}X_n = p^2, \quad C_X(m, n) = 0.$$

## 1.4 Independence

**Definition 1.11.** A stochastic process  $X : \Omega \rightarrow \mathcal{X}^T$  is said to be **independent** if for all finite subsets  $S \subseteq T$ , the finite collection of events  $\{X_s \leq x_s : s \in S\}$  are independent. That is, we have

$$F_{X_S}(x_S) = P(\cap_{s \in S} \{X_s \leq x_s\}) = \prod_{s \in S} P\{X_s \leq x_s\} = \prod_{s \in S} F_{X_s}(x_s).$$

*Remark 6.* Independence of a random process is equivalent to factorization of any finite dimensional distribution function into product of individual marginal distribution functions.

**Example 1.12.** Consider a probability space  $(\Omega, \mathcal{F}, P)$  defined by the sample space  $\Omega = \{H, T\}^{\mathbb{N}}$ , the event space  $\mathcal{F} \triangleq \sigma(E_n : n \in \mathbb{N})$  where  $E_n = \{\omega \in \Omega : \omega_n = H\}$ , and the probability measure  $P : \mathcal{F} \rightarrow [0, 1]$  defined by

$$P(\cap_{i \in F} E_i) = p^{|F|}, \text{ for all finite } F \subseteq \mathbb{N}.$$

Then, we observe that the random sequence  $X : \Omega \rightarrow \{0, 1\}^{\mathbb{N}}$  defined by  $X_n(\omega) \triangleq \mathbb{1}_{E_n}(\omega)$  for all outcomes  $\omega \in \Omega$  and  $n \in \mathbb{N}$ , is independent.

**Definition 1.13.** Two stochastic processes  $X : \Omega \rightarrow \mathcal{X}^{T_1}, Y : \Omega \rightarrow \mathcal{Y}^{T_2}$  are **independent**, if the corresponding event spaces  $\sigma(X), \sigma(Y)$  are independent. That is, for any  $x \in \mathbb{R}^{S_1}, y \in \mathbb{R}^{S_2}$  for finite  $S_1 \subseteq T_1, S_2 \subseteq T_2$ , the events  $A_{S_1}(x) \triangleq \cap_{s \in S_1} X_s^{-1}(-\infty, x_s]$  and  $B_{S_2}(y) \triangleq \cap_{s \in S_2} Y_s^{-1}(-\infty, y_s]$  are independent. That is, the joint finite dimensional distribution of  $X$  and  $Y$  factorizes, and

$$P(A_{S_1}(x) \cap B_{S_2}(y)) = P(A_{S_1}(x))P(B_{S_2}(y)) = F_{X_{S_1}}(x)F_{Y_{S_2}}(y), \quad x \in \mathbb{R}^{S_1}, y \in \mathbb{R}^{S_2}.$$

## 1.5 Filtration

Let  $(\Omega, \mathcal{F}, P)$  be a probability space.

**Definition 1.14.** A net of event spaces denoted  $\mathcal{F}_\bullet = (\mathcal{F}_t \subseteq \mathcal{F} : t \in T)$  is called a **filtration** if the index set  $T$  is totally ordered and the net is nondecreasing, that is  $\mathcal{F}_s \subseteq \mathcal{F}_t$  for all  $s \leq t$ .

**Definition 1.15.** Consider a real-valued random process  $X$  indexed by the ordered set  $T$  on the probability space  $(\Omega, \mathcal{F}, P)$ . The process  $X$  is called **adapted** to the filtration  $\mathcal{F}_\bullet$ , if for each  $t \in T$ , we have  $\sigma(X_t) \subseteq \mathcal{F}_t$  or  $X_t^{-1}(-\infty, x] \in \mathcal{F}_t$  for each  $x \in \mathbb{R}$ .

We will consider any random process  $X : \Omega \rightarrow \mathcal{X}^T$  defined on this probability space with state space  $\mathcal{X} \subseteq \mathbb{R}$  and ordered index set  $T \subseteq \mathbb{R}$  considered as time.

**Definition 1.16.** For the random process  $X : \Omega \rightarrow \mathcal{X}^T$ , we define the event space generated by all random variables until time  $t$  as  $\mathcal{G}_t \triangleq \sigma(X_s, s \leq t)$ .

*Remark 7.* The collection of event spaces  $\mathcal{G}_\bullet = (\mathcal{G}_t : t \in T)$  is a filtration.

**Definition 1.17.** The natural filtration associated with a random process  $X : \Omega \rightarrow \mathcal{X}^T$  is given by  $\mathcal{G}_\bullet = (\mathcal{G}_t : t \in T)$  where  $\mathcal{G}_t \triangleq \sigma(X_s, s \leq t)$ .

*Remark 8.* Any random process  $X$  is adapted to its natural filtration.

*Remark 9.* For a random sequence  $X : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$ , the natural filtration is a sequence  $\mathcal{G}_\bullet = (\mathcal{G}_n : n \in \mathbb{N})$  of event spaces  $\mathcal{G}_n \triangleq \sigma(X_1, \dots, X_n)$  for all  $n \in \mathbb{N}$ .

**Example 1.18.** For a random walk  $S : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  with step size sequence  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  defined by  $S_n \triangleq \sum_{i=1}^n X_i$  for all  $n \in \mathbb{N}$ , the natural filtration of the random walk is identical to that of the step size sequence. That is,  $\sigma(S_1, \dots, S_n) = \sigma(X_1, \dots, X_n)$  for all  $n \in \mathbb{N}$ . This follows from the fact that for all  $n \in \mathbb{N}$ , we can write  $S_j = \sum_{i=1}^j X_i$  and  $X_j = S_j - S_{j-1}$  for all  $j \in [n]$ . That is, there is a bijection between  $(X_1, \dots, X_n)$  and  $(S_1, \dots, S_n)$ .

*Remark 10.* If the random sequence  $X$  is independent, then the random sequence  $(X_{n+j} : j \in \mathbb{N})$  is independent of the event space  $\sigma(X_1, \dots, X_n)$ .

*Remark 11.* If  $X : \Omega \rightarrow \mathcal{X}^T$  is an independent process with the associated natural filtration  $\mathcal{G}_\bullet$ , then for any  $t > s$  and events  $A \in \mathcal{G}_s$ , the random variable  $X_t$  is independent of the event  $A$ . This is just a fancy way of saying  $X_t$  is independent of  $\sigma(X_u, u \leq s)$ . Hence, for any random variable  $Y \in \mathcal{F}_s$ , we have

$$\mathbb{E}[X_t Y | \mathcal{F}_s] = Y \mathbb{E}[X_t].$$

## 1.6 Progressive measurability

For continuous-time processes, where the time  $t$  ranges over an arbitrary index set  $T \subseteq \mathbb{R}$ , the property of being adapted is too weak to be helpful in many situations. Instead, we need to consider measurability of the process as a map  $X : T \times \Omega \rightarrow \mathbb{R}$ . To this end, we first define measurability on the product spaces.

**Definition 1.19.** Let  $\mathcal{S}$  and  $\mathcal{V}$  be two event spaces. The product event space denoted  $\mathcal{S} \otimes \mathcal{V}$  is defined as

$$\mathcal{S} \otimes \mathcal{V} \triangleq \sigma(s \times v : s \in \mathcal{S}, v \in \mathcal{V}).$$

**Definition 1.20.** For a random process  $X : \Omega \rightarrow \mathcal{X}^T$  and any time  $s \in T$ , we can define a stopped process  $X^s : \Omega \rightarrow \mathcal{X}^T$  such that  $X_t^s \triangleq X_{t \wedge s}$  for all  $t \in T$ .

**Definition 1.21.** A process  $X : T \times \Omega \rightarrow \mathbb{R}$  adapted to filtration  $\mathcal{F}_\bullet$  is **progressive** or **progressively measurable**, if stopped process  $X^s$  is  $\mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$  measurable for all  $s \in T$ .

*Remark 12.* Since  $\pi_\Omega \circ (X^{-1}((-\infty, x] \cap (\{t\} \times \Omega))) = X_t^{-1}((-\infty, x]) \in \mathcal{F}_t$ , every progressively measurable process is adapted and jointly measurable.

**Lemma 1.22.** When  $T$  is countable, every adapted process is progressive.

*Proof.* It suffices to show this for countable  $T = \mathbb{N}$ . Let  $X : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$  be a real valued process adapted to filtration  $\mathcal{F}_\bullet$ , and  $X^m$  be a stopped process for  $m \in \mathbb{N}$ . We observe that the inverse map  $(X^m)^{-1}((-\infty, x]) \triangleq \{(n, \omega) : n \leq m, X(n, \omega) \leq x\} = \cup_{n \in [m]} (\{n\} \times X_n^{-1}((-\infty, x])) \in \mathcal{B}([m]) \otimes \mathcal{F}_m$ .  $\square$

**Definition 1.23.** A set  $S \subseteq T \times \Omega$  is said to be **progressive** if its indicator function  $\mathbb{1}_S$  is progressive. Equivalently,  $S \cap ((-\infty, s] \times \Omega) \in \mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$  for all  $s \in T$ .

**Proposition 1.24.** The progressively measurable sets form a  $\sigma$ -algebra.

*Proof.* By definition product event space  $\mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$  is a  $\sigma$ -algebra for all  $s \in T$ . We define the collection

$$\mathcal{G} \triangleq \{S \subseteq T \times \Omega : S \cap ((-\infty, s] \times \Omega) \in \mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s \text{ for all } s \in T\}.$$

We need to show the following three conditions for  $\mathcal{G}$  to be a  $\sigma$ -algebra.

- (i) It is easy to see that  $T \times \Omega \in \mathcal{G}$  since  $(T \times \Omega) \cap ((-\infty, s] \times \Omega) = (-\infty, s] \times \Omega \in \mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$  for all  $s \in T$  by definition.
- (ii) Let  $S \in \mathcal{G}$ , then we will show that  $S^c \in \mathcal{G}$ . Let  $s \in T$ , then using the fact that  $\mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$  is a  $\sigma$ -algebra, it follows that  $S^c \cap ((-\infty, s] \times \Omega) = (S \cap (-\infty, s] \times \Omega)^c \cap ((-\infty, s] \times \Omega) \in \mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$ .
- (iii) Let  $S \in \mathcal{G}^{\mathbb{N}}$  then  $S_n \cap (-\infty, s] \times \Omega \in \mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$  for all  $n \in \mathbb{N}$ , and we will show that  $\bigcup_{n \in \mathbb{N}} S_n \in \mathcal{G}$ . We fix a  $s \in T$ , and the result follows from the distributive property of intersections, and the closure of  $\mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$  under countable unions. □

**Proposition 1.25.** *A stochastic process is progressive iff it is measurable with respect to progressive  $\sigma$ -algebra.*

*Proof.* Let  $X : \Omega \rightarrow \mathcal{X}^T$  be a random process adapted to a filtration  $\mathcal{F}_\bullet$ . Let  $X$  be progressive and fix  $s \in T$  and  $x \in \mathbb{R}$ , then we show that any event generated by the stopped process  $X^s$  is progressive. Indeed, we observe that  $(X^s)^{-1}(-\infty, x] \cap ((-\infty, u] \times \Omega) = (X^{s \wedge u})^{-1}(-\infty, x] \in \mathcal{B}((-\infty, u]) \otimes \mathcal{F}_u$  for all  $u \in T$ .

Conversely, if we assume that any event generated by  $X$  is progressive, then  $X^{-1}(-\infty, x] \cap ((-\infty, s] \times \Omega) = (X^s)^{-1}(-\infty, x] \in \mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$  for all  $s \in T$  and  $x \in \mathbb{R}$ . It follows that  $X$  is progressive. □

**Proposition 1.26.** *Every adapted process with right-continuous sample paths is progressively measurable.*

**Theorem 1.27.** *Every measurable and adapted process has a progressively measurable modification.*