Lecture-05: Stopping Times

1 Stopping Times

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{F}_\bullet = (\mathcal{F}_t \subseteq \mathcal{F} : t \in T)$ be a filtration on this probability space for an ordered index set $T \subseteq \mathbb{R}$ considered as time.

Definition 1.1. A random variable $\tau : \Omega \to T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a **stopping time** with respect to a filtration $\mathcal{F}_\bullet$ if τ is almost surely finite and the event $\{\tau \leqslant t\} \in \mathcal{F}_t$ for all $t \in T$.

Remark 1. Let $\mathcal{F}_\bullet$ be a natural filtration associated with a real-valued time-evolving random process $X : \Omega \to \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$. That is, $\mathcal{F}_t = \sigma(X_s, s \leqslant t)$ for all times $t \in T$.

Remark 2. A stopping time $\tau : \Omega \to T$ for the process X is an almost surely finite random variable such that if we observe the process X sequentially, then the event $\{\tau \leqslant t\}$ can be completely determined by the sequential observation $(X_s, s \leqslant t)$ until time t.

Remark 3. The intuition behind a stopping time is that its realization is determined by the past and present events but not by future events. That is, given the history of the process until time t, we can tell whether the stopping time is less than or equal to t or not. In particular, $\mathbb{E}[\mathbb{1}_{\{\tau \leqslant t\}} \mid \mathcal{F}_t] = \mathbb{1}_{\{\tau \leqslant t\}}$ is either one or zero.

Definition 1.2. For a process $X : \Omega \to \mathcal{X}^T$ and any Borel measurable set $A \in \mathcal{B}(\mathcal{X})$, **first hitting time** to states A by the process X is denoted by $\tau_X^A : \Omega \to T \cup \{\infty\}$, defined as $\tau_X^A \triangleq \inf\{t \in T : X_t \in A\}$.

Example 1.3. Let the process X be adapted to a filtration $\mathcal{F}_\bullet$. Then, we observe that the event $\{\tau_X^A \leqslant t\} = \{X_s \in A \text{ for some } s \leqslant t\} \in \mathcal{F}_t$ for all $t \in T$. It follows that, τ_A^X is a stopping time with respect to filtration $\mathcal{F}_\bullet$ if τ_A^X is finite almost surely.

Theorem 1.4. *Consider an almost surely finite random variable $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ and a filtration $\mathcal{F}_\bullet$ defined on the probability space $(\Omega, \mathcal{F}, P)$. The random variable τ is a* ***stopping time*** *with respect to this filtration $\mathcal{F}_\bullet$ iff the event $\{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$.*

Proof. We first show that if $\{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$, then τ is a stopping time. It follows from the fact that $\{\tau \leqslant n\} = \cup_{m \leqslant n} \{\tau = m\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$.

For the converse, we assume that τ is a stopping time and fix an $n \in \mathbb{N}$. Then $\{\tau \leqslant n\} \in \mathcal{F}_n$ and $\{\tau \leqslant n-1\} \in \mathcal{F}_n$. The result follows from the closure of an event space under complements and intersections, which implies that $\{\tau = n\} = \{\tau \leqslant n\} \setminus \{\tau \leqslant n-1\} \in \mathcal{F}_n$. □

Example 1.5. Consider a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ with the natural filtration $\mathcal{F}_\bullet$ and a measurable set $A \in \mathcal{B}(\mathcal{X})$. If the first hitting time $\tau_X^A : \Omega \to \mathbb{N} \cup \{\infty\}$ for the sequence X to hit set A is almost surely finite, then τ_X^A is a stopping time. For this case, we can write $\{\tau_X^A = n\} = \cap_{k=1}^{n-1} \{X_k \notin A\} \cap \{X_n \in A\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$.

1.1 Properties of stopping time

Lemma 1.6. *Let $\tau_1, \tau_2 : \Omega \to T$ be stopping times on probability space $(\Omega, \mathcal{F}, P)$ with respect to filtration $\mathcal{F}_\bullet$. Then the following hold true.*

*i_* $\min\{\tau_1, \tau_2\}$ *and* $\max\{\tau_1, \tau_2\}$ *are stopping times.*

ii. *If $P\{\tau_1 \in I\} = 1$ and $P\{\tau_2 \in I\} = 1$ for a countable $I \subseteq T$, then $\tau_1 + \tau_2$ is a stopping time.*

Proof. Let $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$ be a filtration, and τ_1, τ_2 associated stopping times.

i. Result follows since the event $\{\min\{\tau_1, \tau_2\} > t\} = \{\tau_1 > t\} \cap \{\tau_2 > t\} \in \mathcal{F}_t$, and the event $\{\max\{\tau_1, \tau_2\} \leqslant t\} = \{\tau_1 \leqslant t\} \cap \{\tau_2 \leqslant t\} \in \mathcal{F}_t$ for any time $t \in T$.

ii. It suffices to show that the event $\{\tau_1 + \tau_2 \leqslant t\} \in \mathcal{F}_t$ for any $t \in I = \mathbb{N}$. We fix $n \in I$, and it follows from the closure of event space $\mathcal{F}_n$ under countable unions and intersection, that $\{\tau_1 + \tau_2 \leqslant n\} = \bigcup_{m \in \mathbb{N}} \{\tau_1 \leqslant n - m, \tau_2 \leqslant m\} \in \mathcal{F}_n$.

□

Lemma 1.7. *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with* i.i.d. *step-sizes $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ having finite $\mathbb{E}|X_1|$. Let $\tau : \Omega \to \mathbb{N}$ be a random variable independent of the step-size sequence such that $\mathbb{E}|\tau| < \infty$. Then,*

$$\mathbb{E}S_\tau = \mathbb{E}X_1 \mathbb{E}\tau.$$

Proof. Recall that the natural filtration of the random walk and the step-sizes are identical, and we denote it by $\mathcal{F}_\bullet$. We know that $P(\cup_{n \in \mathbb{N}} \{\tau = n\}) = 1$ and recall that conditional expectation of S_τ given the discrete random variable τ is given by $\mathbb{E}[S_\tau \mid \sigma(\tau)] = \sum_{n \in \mathbb{N}} \mathbb{E}[S_\tau \mid \tau = n] \mathbb{1}_{\{\tau = n\}}$. Since $S_n = \sum_{i=1}^n X_i$, we obtain from the tower property and linearity of conditional expectation,

$$\mathbb{E}S_\tau = \mathbb{E}[\mathbb{E}[S_\tau | \sigma(\tau)]] = \mathbb{E}[\sum_{n \in \mathbb{N}} \sum_{i=1}^{n} \mathbb{E}[X_i \mid \tau = n] \mathbb{1}_{\{\tau = n\}}].$$

Since the *i.i.d.* random sequence X is independent of random variable τ, we get $\mathbb{E}[X_i \mid \tau = n] = \mathbb{E}X_1$, and it follows that $\mathbb{E}S_\tau = \mathbb{E}X_1 \mathbb{E}[\sum_{n \in \mathbb{N}} n \mathbb{1}_{\{\tau = n\}}] = \mathbb{E}X_1 \mathbb{E}\tau$. □

Lemma 1.8 (Wald). *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with* i.i.d. *step-sizes $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ having finite $\mathbb{E}|X_1|$. Let $\tau : \Omega \to \mathbb{N}$ be a finite mean stopping time adapted to the natural filtration $\mathcal{F}_\bullet$ of the step-size sequence X. Then,*

$$\mathbb{E}S_\tau = \mathbb{E}X_1 \mathbb{E}\tau.$$

Remark 4. We first examine why the proof of Lemma 1.7 breaks down for Lemma 1.8 when τ is a stopping time with respect to natural filtration of X. In the later case, it is not clear what is the value $\mathbb{E}[X_i | \tau = n]$? For example, consider the *i.i.d.* sequence $X \in \{0,1\}^{\mathbb{N}}$ with $P\{X_i = 1\} = p$ and stopping $\tau \triangleq \inf\{n \in \mathbb{N} : X_i = 1\}$ adapted to natural filtration of X. In this case, for $i \leqslant \tau$

$$\mathbb{E}[X_i | \tau = n] = \mathbb{1}_{\{i = n\}} \neq \mathbb{E}X_i = p.$$

However, we do notice that the result somehow *magically* continues to hold, as

$$\mathbb{E}S_\tau = \mathbb{E}[\sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau = n\}}] = 1 = \mathbb{E}X_1 \mathbb{E}\tau = \frac{p}{p}.$$

Proof. Recall that the filtration generated by the random walk and the step-sizes are identical, and denoted by $\mathcal{F}_\bullet$. From the independence of step sizes, it follows that X_n is independent of $\mathcal{F}_{n-1}$. Since τ is a stopping time with respect to random walk S, we observe that $\{\tau \geqslant n\} = \{\tau > n - 1\} \in \mathcal{F}_{n-1}$, and hence it follows that random variable X_n and indicator $\mathbb{1}_{\{\tau \geqslant n\}}$ are independent and $\mathbb{E}[X_n \mathbb{1}_{\{\tau \geqslant n\}}] = \mathbb{E}X_1 \mathbb{E}\mathbb{1}_{\{\tau \geqslant n\}}$. Therefore,

$$\mathbb{E}\sum_{n=1}^{\tau} X_n = \mathbb{E}\sum_{n \in \mathbb{N}} X_n \mathbb{1}_{\{\tau \geqslant n\}} = \sum_{n \in \mathbb{N}} \mathbb{E}X_n \mathbb{E}\left[\mathbb{1}_{\{\tau \geqslant n\}}\right] = \mathbb{E}X_1 \mathbb{E}\left[\sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau \geqslant n\}}\right] = \mathbb{E}[X_1]\mathbb{E}[\tau].$$

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it by the application of dominated convergence theorem. □

1.2 Stopping time σ-algebra

We wish to define an event space consisting information of the process until a random time τ. For a stopping time $\tau : \Omega \to T$, what we want is something like $\sigma(X_s : s \leqslant \tau)$. But that doesn't make sense, since the random time τ is a random variable itself. When τ is a stopping time, the event $\{\tau \leqslant t\} \in \mathcal{F}_t$. What makes sense is the set of all events whose intersection with $\{\tau \leqslant t\}$ belongs to the event subspace $\mathcal{F}_t$ for all $t \geqslant 0$.

Definition 1.9. For a stopping time $\tau : \Omega \to T$ adapted to the filtration $\mathcal{F}_\bullet$, the **stopping time σ-algebra** is defined

$$\mathcal{F}_\tau \triangleq \{A \in \mathcal{F} : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t, \text{ for all } t \in T\}.$$

Proposition 1.10. *The collection of events $\mathcal{F}_\tau$ is a σ-algebra.*

Proof. It suffices to verify the following three conditions.

(i) Since τ is a stopping time, it follows that $\Omega \in \mathcal{F}_\tau$.

(ii) Let $A \in \mathcal{F}_\tau$, then $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ and we can write $A^c \cap \{\tau \leqslant t\} = \{\tau \leqslant t\} \setminus (A \cap \{\tau \leqslant t\}) \in \mathcal{F}_t$.

(iii) From closure of $\mathcal{F}_t$ under countable unions, it follows that $\mathcal{F}_\tau$ is closed under countable unions.

□

Remark 5. Informally, the event space $\mathcal{F}_\tau$ has information up to the random time τ. That is, it is a collection of measurable sets that are determined by the process until time τ.

Remark 6. Any measurable set $A \in \mathcal{F}$ can be written as $A = (A \cap \{\tau \leqslant t\}) \cup (A \cap \{\tau > t\})$. All such sets A such that $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ for all $t \in T$ is a member of the stopped σ-algebra. Therefore, we note that any event $A \in \mathcal{F}_\tau$ does not guarantee that $A \cap \{\tau > t\} \in \mathcal{F}_t$. Otherwise, $\mathcal{F}_\tau = \mathcal{F}$.

Lemma 1.11. *Let τ, τ_1, τ_2 be stopping times, and $X : \Omega \to \mathcal{X}^T$ a random process, all adapted to a filtration $\mathcal{F}_\bullet$. Then, the following are true.*

i_ *If $\tau_1 \leqslant \tau_2$ almost surely, then $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.*

ii_ *$\sigma(\tau) \subseteq \mathcal{F}_\tau$, and $\sigma(X_\tau) \subseteq \mathcal{F}_\tau$.*

Proof. Recall, that for any $t \geqslant 0$, we have $\{\tau \leqslant t\} \in \mathcal{F}_t$.

i_ From the hypothesis $\tau_1 \leqslant \tau_2$ a.s., we get $\{\tau_2 \leqslant t\} \subseteq \{\tau_1 \leqslant t\}$ a.s., where both events belong to $\mathcal{F}_t$ since they are stopping times. The result follows since for any $A \in \mathcal{F}_{\tau_1}$ and $t \in T$, we can write $A \cap \{\tau_2 \leqslant t\} = A \cap \{\tau_1 \leqslant t\} \cap \{\tau_2 \leqslant t\} \in \mathcal{F}_t$.

ii_ Any event $A \in \sigma(\tau)$ is generated by inverse images $\{\tau \leqslant s\}$ for $s \in \mathbb{R}$. Indeed $\{\tau \leqslant s\} \in \mathcal{F}_\tau$ since $\{\tau \leqslant s\} \cap \{\tau \leqslant t\} = \{\tau \leqslant s \wedge t\} \in \mathcal{F}_t$, for all $t \in T$.

The events of the form $\{X_\tau \leqslant x\}$ for real $x \in \mathbb{R}$ generate the event subspace $\sigma(X_\tau)$, and event $\{X_\tau \leqslant x\} \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ for all $t \in T$. This implies that $\sigma(X_\tau) \subseteq \mathcal{F}_\tau$.

□

Definition 1.12. Consider a process $X : \Omega \to \mathcal{X}^T$ and its natural filtration $\mathcal{F}_\bullet$, and a stopping time $\tau : \Omega \to T$ with respect to X, then the stopped process $(X_{\tau \wedge t} : t \in T)$ is defined for all $t \in T$ by

$$X_{\tau \wedge t} \triangleq X_t \mathbb{1}_{\{t \leqslant \tau\}} + X_\tau \mathbb{1}_{\{t > \tau\}}.$$

Lemma 1.13. *Let $\mathcal{F}_\bullet$ be the natural filtration associated with the process $X : \Omega \to \mathcal{X}^T$, and τ be an associated stopping time. Let $\mathcal{H} \triangleq \sigma(X_{\tau \wedge t}, t \in T)$ be the event space generated by the stopped process $(X_{\tau \wedge t} : t \in T)$ and $\mathcal{F}_\tau$ be the stopping-time event space. Then $\mathcal{F}_\tau = \mathcal{H}$ for T discrete.*

Proof. Let $A \in \mathcal{H}$, then we have $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ for any $t \in T$, and hence $\mathcal{H} \subseteq \mathcal{F}_\tau$.

For the converse, we assume $T = \mathbb{N}$ and recall that for any stopping time τ, we have $\cup_{k \in \mathbb{N}} (A \cap \{\tau = k\}) = A$. To show that $\mathcal{F}_\tau \subseteq \mathcal{H}$, it suffices to show that for any $A \in \mathcal{F}_\tau$ we have $A \cap \{\tau = k\} \in \mathcal{H}$ for all $k \in \mathbb{N}$. We will show this by induction on $k \in \mathbb{N}$.

$k = 1$: We take any $A \in \mathcal{F}_\tau$, then $A \cap \{\tau = 1\} \in \mathcal{F}_1 \subseteq \mathcal{H}$ since $\tau \geqslant 1$ almost surely.

$k > 1$: We assume that the induction hypothesis holds for some $k-1 \in \mathbb{N}$. For any $A \in \mathcal{F}_\tau$, we have $A \cap \{\tau = k\} \in \mathcal{F}_k = \sigma(X_1, \ldots, X_k)$. Further, $\{\tau = k\} = \{\tau = k\} \cap \{\tau \geqslant k\}$, and therefore, we can write

$$\mathbb{1}_{A \cap \{\tau = k\}} = f(X_1, \ldots, X_k) \mathbb{1}_{\{\tau \geqslant k\}} = f(X_{\tau \wedge 1}, \ldots, X_{\tau \wedge k})(1 - \mathbb{1}_{\{\tau \leqslant k-1\}}) \in \mathcal{H}.$$

This implies that $A \cap \{\tau = k\} \in \mathcal{H}$, and hence $\mathcal{F}_\tau \subseteq \mathcal{H}$.

□