Lecture-08: Distribution and renewal functions

1 Convolution of distribution functions

Definition 1.1. For two distribution functions $F, G : \mathbb{R} \to [0,1]$ the convolution of F and G is a distribution function $F * G : \mathbb{R} \to [0,1]$ defined as

$$(F * G)(x) \triangleq \int_{y \in \mathbb{R}} F(x-y) dG(y),\ x \in \mathbb{R}.$$

Remark 1. Verify that $F * G$ is indeed a distribution function. That is, the function $(F * G)$ is

(a) right continuous, i.e. $\lim_{x_n \downarrow x} (F * G)(x_n)$ exists,

(b) non-decreasing, i.e. $(F * G)(z) \geqslant (F * G)(x)$ for all $z \geqslant x$,

(c) having left limit of zero and right limit of unity, i.e. $\lim_{x \to -\infty} (F * G)(x) = 0, \lim_{x \to \infty} (F * G)(x) = 1$.

Remark 2. Verify that convolution is a symmetric and bi-linear operator. That is, for any distribution functions (F, G) and $(F_i : i \in [n]), (G_j : j \in [m])$ and vectors $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m$, we have

$$F * G = G * F, \qquad \Big(\sum_{i \in [n]} \alpha_i F_i\Big) * \Big(\sum_{j \in [m]} \beta_j G_j\Big) = \sum_{i \in [n]} \sum_{j \in [m]} \alpha_i \beta_j (F_i * G_j).$$

Lemma 1.2. *Let X and Y be two independent random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with distribution functions F and G respectively, then the distribution of $X + Y$ is given by $F * G$.*

Proof. The distribution function of sum $X + Y$ is given by $H : \mathbb{R} \to [0,1]$ where for any $z \in \mathbb{R}$,

$$H(z) = \mathbb{E}\mathbb{1}_{\{X+Y \leqslant z\}} = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{X+Y \leqslant z\}} | \sigma(Y)]] = \mathbb{E}[F(z-Y)] = \int_{y \in \mathbb{R}_+} F(z-y) dG(y).$$

□

Definition 1.3. Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be an independent random sequence defined on the probability space $(\Omega, \mathcal{F}, P)$ with distribution function F, then the distribution of $S_n \triangleq \sum_{i=1}^n X_i$ is given by $F_n \triangleq F_{n-1} * F$ for all $n \geqslant 2$ and $F_1 = F$.

Lemma 1.4. *Consider a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with* i.i.d. *inter-renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having common distribution $F : \mathbb{R}_+ \to [0,1]$. The distribution function of nth renewal instant $S_n \triangleq \sum_{i=1}^n X_i$ is given by $P\{S_n \leqslant t\} = F_n(t)$ for all $t \in \mathbb{R}$, where F_n is n-fold convolution of the distribution function F.*

Remark 3. The distribution function F_n is computed inductively as $F_n = F_{n-1} * F$, where $F_1 = F$.

Example 1.5 (Poisson process). For a renewal sequence S with the common distribution for *i.i.d.* inter-renewal times being $F(x) = 1 - e^{-\lambda x}$ for $x \in \mathbb{R}_+$, the distribution of nth renewal instant is

$$F_n(t) = \int_0^t \frac{\lambda^n s^{n-1}}{(n-1)!} e^{-\lambda s} ds.$$

Corollary 1.6. *The distribution function of nth arrival instant S_n for delayed renewal process is $G * F_{n-1}$.*

Corollary 1.7. *The distribution function of counting process $N^D : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ for the delayed renewal process is*

$$P\left\{N_t^D = n\right\} = P\{S_n \leqslant t\} - P\{S_{n+1} \leqslant t\} = (G * F_{n-1})_t - (G * F_n)_t. \tag{1}$$

2 Renewal functions

Definition 2.1. Mean of the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is called the **renewal function** denoted by $m : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $m_t = \mathbb{E}[N_t]$ for all $t \in \mathbb{R}_+$.

Proposition 2.2. *Renewal function m for a renewal process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with distribution of renewal instants $F_n \triangleq F_{S_n}$ for each $n \in \mathbb{N}$, can be written as $m_t = \sum_{n\in\mathbb{N}} F_n(t)$.*

Proof. Using the inverse relationship between counting process and the arrival instants, we can write

$$m_t = \mathbb{E}[N_t] = \sum_{n\in\mathbb{N}} P\{N_t \geqslant n\} = \sum_{n\in\mathbb{N}} P\{S_n \leqslant t\} = \sum_{n\in\mathbb{N}} F_n(t).$$

□

Example 2.3 (Poisson process). For a renewal sequence S with the common distribution for *i.i.d.* inter-renewal times being $F(x) = 1 - e^{-\lambda x}$ for $x \in \mathbb{R}_+$, the renewal function is

$$m_t = \sum_{n\in\mathbb{N}} F_n(t) = \int_0^t \lambda \Big(e^{-\lambda s} \sum_{n\in\mathbb{Z}_+} \frac{(\lambda s)^n}{s!} \Big) ds = \int_0^t \lambda ds = \lambda t.$$

Corollary 2.4. *The renewal function m_D for a delayed renewal process $N_D : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ with distribution G for first inter-renewal times and F for other inter-renewal times, is given by $m_D = G + G * m$.*

Proof. We can write the renewal function for the delayed renewal process as

$$m_t^D = \mathbb{E}N_t^D = \sum_{n\in\mathbb{N}} (G * F_{n-1})_t = G_t + (G * m)_t. \tag{2}$$

□

Remark 4. If $G = F$, then $m = F + F * m$

3 Laplace transform of distribution functions and renewal functions

Definition 3.1. For a distribution function $F : \mathbb{R} \to [0,1]$ the Laplace transform $\mathcal{L}(F)$ is a map $\tilde{F} : \mathbb{C} \to \mathbb{C}$ defined

$$\tilde{F}(s) \triangleq \int_{y\in\mathbb{R}} e^{-sy} dF(y) = \mathbb{E}[e^{-sX}],$$

where s lies in the region such that $\left|\tilde{F}(s)\right| < \infty$, and X is a random variable with distribution F.

Lemma 3.2. *The Laplace transform of convolution of two distribution functions is product of Laplace transform of individual distribution functions.*

Proof. Let $F, G : \mathbb{R} \to [0,1]$ be two distribution functions such that $\mathcal{L}(F) = \tilde{F}$ and $\mathcal{L}(G) = \tilde{G}$, then

$$\mathcal{L}(F * G)(s) = \int_{x\in\mathbb{R}} e^{-sx} \int_{y\in\mathbb{R}} dF(x-y) dG(y) = \int_{y\in\mathbb{R}} e^{-sy} dG(y) \int_{x-y\in\mathbb{R}} e^{-s(x-y)} dF(x-y) = \tilde{F}(s)\tilde{G}(s).$$

□

Corollary 3.3. *Let X and Y be two independent random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with Laplace transform of distribution functions $\tilde{F}$ and $\tilde{G}$ respectively, then the Laplace transform of the distribution of $X + Y$ is given by $\tilde{F}\tilde{G}$.*

Corollary 3.4. *Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be an independent random sequence defined on the probability space $(\Omega, \mathcal{F}, P)$ with the Laplace transform of the distribution function given by $\tilde{F}$, then the Laplace transform of the distribution of $S_n \triangleq \sum_{i=1}^n X_i$ is given by $\mathcal{L}(F_n) = (\tilde{F})^n$.*

Corollary 3.5. *We denote the Laplace transform for the inter-arrival time distribution F by $\mathcal{L}(F) = \tilde{F}$, then the Laplace transform of the renewal function m is given by*

$$\tilde{m}(s) = \frac{\tilde{F}(s)}{1 - \tilde{F}(s)}, \qquad \Re\left\{\tilde{F}(s)\right\} < 1.$$

Example 3.6 (Poisson process). The Laplace transform of an exponential distribution $F(x) = 1 - e^{-\lambda x}$ for $x \in \mathbb{R}_+$ is given by $\tilde{F}(s) = \frac{\lambda}{\lambda+s}$ for $\Re(s) > -\lambda$. For a renewal sequence S with the common distribution for *i.i.d.* inter-renewal times being the exponential distribution F, the Laplace transform for the renewal function is

$$\tilde{m}(s) = \frac{\tilde{F}(s)}{1 - \tilde{F}(s)} = \frac{\lambda}{s}, \qquad \Re(s) > -\lambda.$$

The Laplace transform for the distribution F_n is given by

$$\tilde{F}_n(s) = \left(1 + \frac{s}{\lambda}\right)^{-n}, \qquad \Re(s) > -\lambda.$$

Lemma 3.7. *Let the Laplace transforms for the distributions of the first inter-arrival time and the subsequent inter-arrival times be denoted by $\tilde{G} = \mathcal{L}(G)$ and $\tilde{F} = \mathcal{L}(F)$ respectively, then the Laplace transform of the renewal function m_D for the delayed renewal process is*

$$\tilde{m}_D(s) = \frac{\tilde{G}(s)}{1 - \tilde{F}(s)}, \quad \Re\left\{\tilde{F}(s)\right\} < 1. \tag{3}$$

Proposition 3.8. *For renewal process with $\mathbb{E}X_n > 0$, the renewal function is bounded for all finite times.*

Proof. Since we assumed that $P\{X_n = 0\} < 1$, it follow from continuity of probabilities that there exists $\alpha > 0$, such that $P\{X_n \geqslant \alpha\} = \beta > 0$. We can define bivariate random variables

$$\bar{X}_n = \alpha 1_{\{X_n \geqslant \alpha\}} \leqslant X_n.$$

Note that since X_i's are *i.i.d.* , so are $\bar{X}_i$'s. Each $\bar{X}_i$ takes values in $\{0, \alpha\}$ with probabilities $1 - \beta$ and β respectively. Let $\bar{N}_t$ denote the renewal process with inter-arrival times $\bar{X}_n$, with arrivals at integer multiples of α. Then for all sample paths, we have

$$N_t = \sum_{n \in \mathbb{N}} \mathbb{1}_{\left\{\sum_{i=1}^n X_i \leqslant t\right\}} \leqslant \sum_{n \in \mathbb{N}} \mathbb{1}_{\left\{\sum_{i=1}^n \bar{X}_i \leqslant t\right\}} = \bar{N}_t.$$

Hence, it follows that $\mathbb{E}N_t \leqslant \mathbb{E}\bar{N}_t$, and we will show that $\mathbb{E}\bar{N}_t$ is finite. We can write the joint event of number of arrivals n_i at each arrival instant in $i\alpha$ for $i \in \{0, \ldots, k-1\}$, as

$$\bigcap_{i=0}^{k-1} \{\bar{N}_{i\alpha} = n_i\} = \{X_1 < \alpha\} \bigcap_{i=0}^{k-1} \{X_{n_i+1} \geqslant \alpha\} \bigcap_{i=0}^{k-1} \bigcap_{j=2}^{n_i} \{X_{n_{i-1}+j} < \alpha\}.$$

It follows that the joint distribution of number of arrivals at first k arrival instants is

$$P\left(\bigcap_{i=0}^{k-1} \{\bar{N}_{i\alpha} = n_i\}\right) = (1-\beta) \prod_{i=0}^{k-1} (\beta)(1-\beta)^{n_i - 1}.$$

It follows that the number of arrivals is independent at each arrival instant $k\alpha$ and geometrically distributed with mean $1/\beta$ and $(1-\beta)/\beta$ for $k \in \mathbb{N}$ and $k = 0$ respectively. Thus, for all $t \geqslant 0$,

$$\mathbb{E}N_t \leqslant \mathbb{E}\bar{N}_t \leqslant \frac{\lceil \frac{t}{\alpha} \rceil}{\beta} \leqslant \frac{\frac{t}{\alpha} + 1}{\beta} < \infty. \tag{4}$$

□

Corollary 3.9. *For delayed renewal process with $\mathbb{E}X_n > 0$, the renewal function is bounded at all finite times.*